Milnor's Invariants of Pure 3-Braids

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Abstract

The Milnor's invariants of a pure braid's closure provide information on how complex the braid is (specifically, how difficult it is to pull apart or untangle). Murasugi previously showed that all 3-braids are conjugate to a 3-braid of one of three forms. Since any two braids that are conjugate have the same Milnor's invariants, we can completely describe the Milnor's invariants of pure braids on 3 strands by considering the Milnor's invariants of braids in the three classes outlined by Murasugi. This paper explores two key questions: when are 3-braids in these three classes pure, and what do their closures' Milnor's invariants look like?

1 Introduction

Informally, one could think of an n-braid as an intertwining of n disjoint strands. A braid can be thought of as a parametrization of n points being pushed around on a a disk, with the strands representing permutations of the points over time. Thus, no strand on a braid loops back on itself; all strands continually move downward. Since we can think of a braid as a subset of the product of the unit interval and a disk, a pure braid on n strands (n-braid) is a braid whose strands start and end in the same position on this disk. Formally, a pure braid is defined as follows.

Definition 1. *Let D be the unit disk, I the unit interval, and $\{p_1, p_2, \ldots, p_k\}$ be n points in the interior of D. An n-component pure braid is a smooth, proper embedding $\sigma : \bigsqcup_n I \to D \times I$ such that*

$$\sigma|_{I_i(0)} = \{p_i\} \times \{0\}$$
$$\sigma|_{I_i(1)} = \{p_i\} \times \{1\}$$
$$\sigma|_{I_i(x)} = \{q_{i,x}\} \times \{x\}$$

where $q_{i,x} \in D$.

The set of all braids on n strands forms a group under the operation of stacking, as does the set of all pure braids, denoted PB_n. Then, we can consider elements of this group to be conjugate.

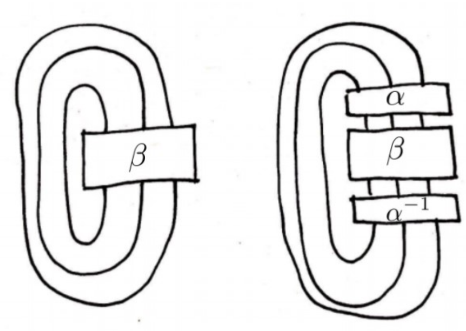
Definition 2. *Two n-braids β, β' are conjugate if, for some n-braid α, $\beta' = \alpha\beta\alpha^{-1}$.*

Murasugi [Mur74] states that all 3-braids are conjugate to one of the following classes of braids.

1. $h^d\sigma_1\sigma_2^{-a_1}...\sigma_1\sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$
2. $h^d\sigma_2^m$ with $m \in \mathbb{Z}$
3. $h^d\sigma_1^m\sigma_2^{-1}$ with $m \in \{1, 2, 3\}$

where $h = (\sigma_1\sigma_2)^3$

We can take the closure of a pure braid, and this closure is a link. As it turns out, if two braids are conjugate, their closures will be the same link. This can be visualized as follows, where the second link can be deformed into the first.



In other words, we can completely describe the Milnor's invariants of pure braids on 3 strands by describing the Milnor's invariants of the three forms of braids Murasugi defined.

Milnor's invariants are a concordance invariant for links that provide meaningful information about how components are linked. Milnor's invariants can be thought of as higher-order versions of linking number (that is, the integer-valued *linking numbers* are the first Milnor's invariants one can take, and when these are insufficient, there are Milnor's invariants of higher weight). In this paper, we use surface systems to compute Milnor's invariants, which we'll briefly outline before presenting any computations.

Applying a link concordance invariant to closures of pure braids can help us in understanding when a link is concordant to a closure of a pure braid; if a three-component link has Milnor's invariants that are not possible for the closure of a pure braid, this obstructs the link from being concordant to the closure of a pure braid. We won't discuss concordance much in this paper, but it's an important equivalence relation used to describe knots and links.

2 When are 3-braids pure?

Pure braids are interesting objects of study. Since we want to know what the Milnor's invariants of pure braids on 3 strands look like, we first need to identify when a braid in one of the three classes outlined by Murasugi is pure.

2.1 First class

To better understand braids of the form $h^d\sigma_1\sigma_2^{-a_1}...\sigma_1\sigma_2^{-a_n}$, we can consider the permutations these braids represent. That is, we can construct a homomorphism $\phi : B_3 \to S_3$ and work with permutations in S_3 instead of $3-$braids. Then, a braid β in B_3 is pure if and only if $\phi(\beta) = e_{S_3}$. This is the main proof strategy in this section; it can also be used elsewhere, but is most necessary here since this class of braids is the largest and most complicated.

Proposition 1. *Let β be a braid of the form $h^d\sigma_1\sigma_2^{-a_1}...\sigma_1\sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$ where $a_1, a_2, ..., a_n$ are all odd. Then, β is pure if and only if n is divisible by 3.*

Proof. Since h is pure, h^d is pure for any integer d since the composition of two pure braids is also pure. Thus, we will only need to consider the portions of a braid below h^d.

Let a_i be odd. Then, the section $\sigma_1\sigma_2^{-a_i}$ of β maps to the following permutation in S_3.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

It follows that $\sigma_1\sigma_2^{-a_1}...\sigma_1\sigma_2^{-a_n}$ maps to this permutation to the nth power in S_3. That is,

$$\sigma_1\sigma_2^{-a_1}...\sigma_1\sigma_2^{-a_n} \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^n$$

Taking $n = 1, 2, 3$, we see that this is a cycle of length 3.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Then, for a braid β with $a_1, a_2, ..., a_n$ all odd integers, β will map to one of the three permutations above depending on n.

If $n = 0 (mod 3)$, then

$$\phi(\beta) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

If $n = 1 (mod 3)$, then

$$\phi(\beta) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

And, if $n = 2 (mod 3)$, then

$$\phi(\beta) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Thus, for a braid β of the form $h^d \sigma_1 \sigma_2^{-a_1} ... \sigma_1 \sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$, if $a_1, a_2, ..., a_n$ are all odd and n is divisible by 3 then β is pure. If $a_1, a_2, ..., a_n$ are all odd and n is not divisible by 3 then β is not pure. □

Proposition 2. *Let β be a braid of the form $h^d \sigma_1 \sigma_2^{-a_1} ... \sigma_1 \sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$ where $a_1, a_2, ..., a_n$ are all even. Then, β is pure if and only if n is divisible by 2.*

Proof. Let $a_1, a_2, ..., a_n$ be even integers. Then,

$$\sigma_1 \sigma_2^{-a_1} ... \sigma_1 \sigma_2^{-a_n} \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}^n$$

Where, for odd n,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and for even n,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

This is a cycle of length 2. It follows that that β is pure if and only if n is even. □

By considering braids as elements of S_3, we see that the behavior of these subsets of the first class of braids is determined only by n (the number of indices $a_1, ..., a_n$) and whether these indices are even or odd, though not their actual numerical value. Then, we can use a similar proof strategy of considering braids as permutations in S_3 then identifying a finite cycle that arises when constructing braids of a particular form (since every braid will be constructed by composing the two permutations above some number of times). This provides us additional results such as:

Lemma 2.1. *Let $a_1, a_2, ..., a_n$ be integers with alternating parity (that is, let a_i for odd i and a_i for even i have opposite parity). Then, β is pure if and only if 4 divides n.*

Lemma 2.2. *Let $a_2, a_3, ..., a_n$ be integers of the same parity (either even or odd) and let a_1 have opposite parity. Then, β is not pure.*

Additionally, we can combine braids of any of these forms (by stacking) any number of times to arrive at additional pure braids.

Corollary 2.2.1. *Let α, β be pure braids (for example, braids of one of the previously mentioned forms) and let $\alpha = h^d\alpha'$ and $\beta = h^d\beta'$. Then, a braid of the form $h^d\alpha'\beta'$ will also be pure.*

Proof. This follows naturally from the fact that compositions of pure braids are also pure. □

As it turns out, it's not too difficult to find pure braids in this first class of braids; there are numerous examples. It is, however, difficult to determine whether braids of the forms listed above are the only pure braids in Murasugi's first class of braids or if there exist more (and if so, how many more?).

2.2 Second class

Theorem 2.3. *Let β be a braid of the form $h^d\sigma_2^m$ with $m \in \mathbb{Z}$. For any $d \in \mathbb{Z}$, β is a pure braid if and only if m is even.*

Proof. Let β be a braid of the form $h^d\sigma_2^m$ with $m \in \mathbb{Z}$. Since h is pure, h^d will also be pure for any integer d. Thus, we will only need to consider the portion of β below h^d. That is, we will only consider σ_2^m.

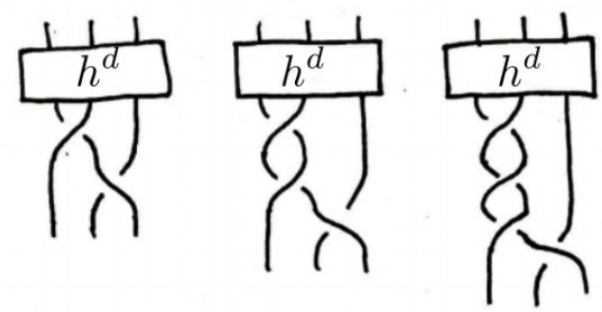
Recall that σ_2 is the generator that permutes the second and third strand. σ_2 is its own inverse, since permuting the strands a second time returns them to their original location; this is a cycle of length 2. Then, for an even integer m, σ_2^m is pure, and for an odd integer m, σ_2^m is not pure.

One could also show this using the proof method from the previous section (constructing a homomorphism to S_3 and showing that this finite cycle exists in S_3). □

2.3 Third class

Lemma 2.4. *Let β be a braid of the form $h^d\sigma_1^m\sigma_2^{-1}$ with $m \in \{1, 2, 3\}$. Then, β is not pure.*

Proof. This class of braids is small enough that we can draw all the braids of this form and verify that none of them are pure.



□

3 Milnor's Invariants

All Milnor's invariants taken in this paper are computed using surface systems, which Cochran [Coc90] formally defines as follows. He begins by introducing a system for indexing the curves generated by this process (n-bracketing), then defining a surface system for a link L.

Definition 3. *The set of n-bracketings (in m variables)* B_n *is given inductively by:*

- $B_1 = \{x, y, z...\}$ *and*
- $B_n = \{(\sigma, \omega) \mid \sigma \in B_k, \omega \in B_{n-k}, 1 \leq k \leq n-1\}$

Definition 4. *A surface system of length n for L is a pair* $(C, \mathcal{V})$ *of sets satisfying:*

- *There exists a coherent subset S of* $\cup B_i$ *such that if* $\omega(\sigma) < n$ *then* $\sigma \in S$.
- $\mathcal{V}$ *is a set of compact, oriented, transversly indersecting, 2-dimensional (possibly empty) submanifolds* $V(\sigma)$ *of* $E(L)$, *bijectively indexed by* $\sigma \in S$ *such that* $V(\alpha, \beta)$ *is* $V(\beta, \alpha)$ *with the opposite orientation.*
- *C is a set of closed, oriented (possibly empty) 1-dimensional submanifolds of E(L) containing the longitudes c(x),c(y),c(z),... and all of whose other elements are bijectively indexed by* (β, α) *where* β, α *are in S. Specifically, C is the set consisting of the longitudes of L together with all* $c(\beta, \alpha)$ *where* $c(\beta, \alpha)$ *is* $V(\beta) \cap V(\alpha)$. *By convention,* $V(\alpha) \cap V(\alpha)$ *is empty. These intersections are oriented according to the convention that the ordered triple (orientation of* $c(\beta, \alpha)$, *positive normal to* $V(\beta)$, *positive normal to* $V(\alpha)$) *be the chosen orientation of* S^3. *These positive normals shall be chosen so that the ordered pairs (orientation of the surface, positive normal) give the ambient orientation. We shall let* $c^+(\beta, \alpha)$ *denote either a (+,+), (+,-),(-,+), or a (-,-) push-off of* $c(\beta, \alpha)$ *with respect to*

$V(\beta)$ and $V(\alpha)$ such that the interior of the annulus spanning c and c^+ misses all elements of C. Define $c^+(x)$ to be $c(x)$ and similarly for other longitudes. Since $c(\beta,\alpha) = -c(\alpha,\beta)$, it is required that these push-offs satisfy $c^+(\beta,\alpha) = -c^+(\alpha,\beta)$.

- *$\delta V(\sigma) = c^+(\sigma)$.*
- *Suppose $w(\alpha) + w(\beta) \leq n$; then, $c(\sigma) \cap V(\beta)$ is empty unless $c(\sigma) \subset V(\beta)$. Thus, $c(\sigma) \cap c(\beta)$ is empty unless the images of the curves coincide. Furthermore, it is convenient to impose the condition that if $w(\beta) < n$ then each component of $\delta V(\beta)$ is the entire boundary of a component of $V(\beta)$.*

That is, we can label the components of a closure of a braid as $c(x), c(y)$, and $c(z)$. Then, we can denote surfaces bounded by each component $V(x), V(y), V(z)$ (respectively) and the curves resulting from taking intersections of these surfaces as $c(xy), c(yz), c(xz)$ etc. This is an iterative process; we can continue generating surfaces bounded by curves and taking their intersections until we arrive at a pair of curves with a non-zero linking number. This is called the first *non-vanishing* Milnor's invariant of the link.

Note: For a curve $c(x)$, Cochran references its positive push off of the surfaces it results from, which he denotes $c^+(x)$. The surfaces bounded by pure braids are relatively simple (essentially flat disks), so taking the push off a surface is simply lifting the curve off the page in the positive direction (either towards or away from the viewer, depending on the choice of orientation for the surfaces).

4 Milnor's invariants of pure 3-braids

Armed with some examples of pure 3-braids, we can now describe their Milnor's invariants. What do the Milnor's invariants of these pure braids look like? What weight must they be?

4.1 First class

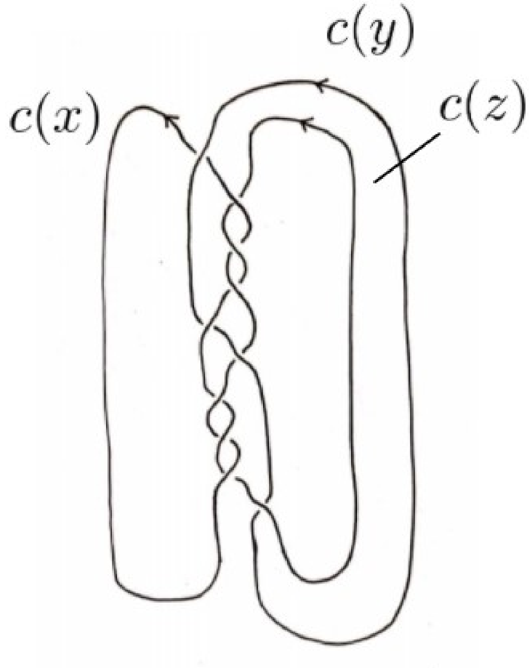
Proposition 3. *There exists a pure braid of the form $h^d \sigma_1 \sigma_2^{-a_1} ... \sigma_1 \sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$ with first non-vanishing Milnor's invariants of weight greater than 2 (linking number is not sufficient for describing first non-vanishing Milnor's invariants of this class of braids).*

Proof. Consider the braid:

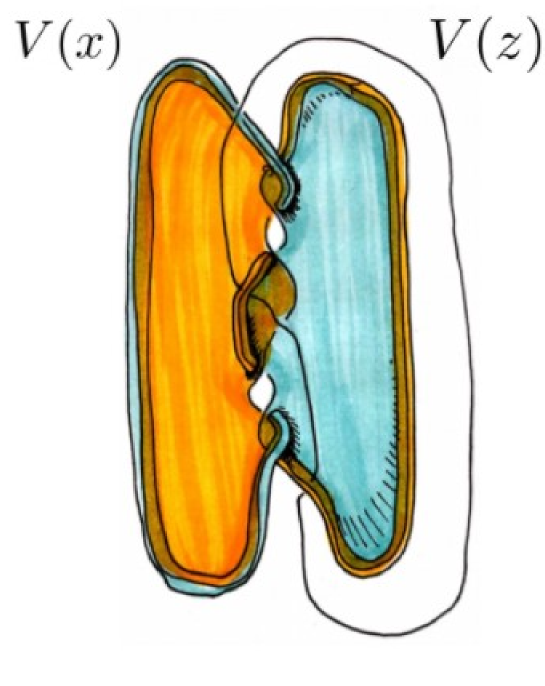
$$\sigma_1\sigma_2^{-3}\sigma_1\sigma_2^{-1}\sigma_1^3\sigma_2^{-1}$$

This braid is an element of the first class of braids outlined by Murasugi, as it can be written $h^0\sigma_1\sigma_2^{-3}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^0\sigma_1\sigma_2^0\sigma_1\sigma_2^0\sigma_2^{-1}$.

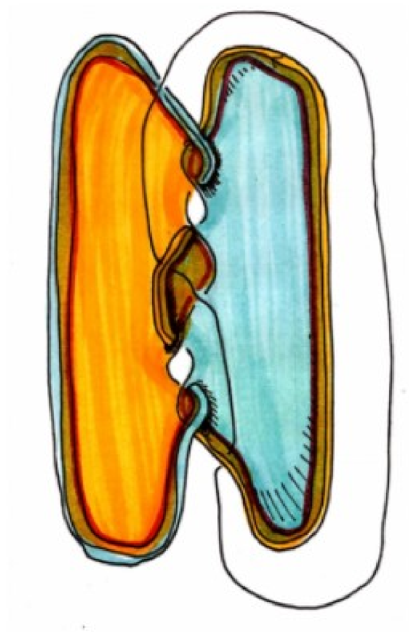
We can draw its closure and verify that it is both non-trivial and has vanishing weight two Milnor's invariants (that is, the linking number between each pair of components is zero). This makes it a good candidate for a link with higher-order non-vanishing Milnor's invariants, so we can orient each component as follows and generate surfaces bounded by each pair of components.



The surfaces $V(x), V(z)$ bounded by $c(x), c(z)$, respectively, are:



And we can identify two curves of intersection $c(xz)$ (where $V(x), V(z)$ intersect), shown in red below.



Then, we find that these curves of intersection are linked with the component $c(y)$ in a non-trivial way. That is,

$$\bar{\mu}(123) = \ell k(c(xz), c(y)) = 2.$$

We can take the linkings $\ell k(c(xy), c(z))$ and $\ell k(c(yz), c(x))$ in a similar fashion, and these will also be equal to 2. Thus, there exists a pure braid of the form $h^d \sigma_1 \sigma_2^{-a_1} ... \sigma_1 \sigma_2^{-a_n}$ with $a_i \geq 0$ and some $a_i > 0$ with first non-vanishing Milnor's invariants of weight 3.

$\square$

In fact, since linking number is additive and we can create new braids by stacking others in this class (i.e., copies of this braid), there are actually multiple examples of braids of this form with first non-vanishing Milnor's invariants of weight 3. Since braids of this form are constructed by stacking various combinations of σ_1 and σ_2^{-1}, it's entirely possible that weight 3 is the highest possible weight for the first non-vanishing Milnor's invariants of a pure braid in this class of 3-braids.

Conjecture 1. *Let β be a braid of the form (class one) such that all weight 2 Milnor's invariants vanish. Then, β will have first non-vanishing Milnor's invariant of weight 3.*

Brunnian braids on 3 strands form a subgroup of the pure braids on 3 strands. They're interesting objects of study when trying to find braids in this class with first non-vanishing Milnor's invariants of weight greater than 2.

Conjecture 2. *Let β be a brunnian braid of the form (class one). Then, β cannot have first non-vanishing Milnor's invariants of weight 2. Furthermore, the first non-vanishing Milnor's invariants of β will be of weight 3.*

4.2 Shorthand for Milnor's invariants of weight 2

Let β be a 3-braid and let $\hat{\beta}$ be the closure of β: a link. Then, denote the first, second, and third component of $\hat{\beta}$ (from the first, second, and third strands of β) x, y, and z. Then, we will define a function $L : \mathcal{B}_3 \to \mathbb{Z}^3$ as follows.

$$L(\sigma) = (\ell k(x,y), \ell k(y,z), \ell k(x,z))$$

We will use this notation in the next proof to describe the linking numbers of pairs of components in the closure of a 3-braid. This makes it easier to recognize when all weight 2 Milnor's invariants of the closure of a braid σ vanish, as $L(\sigma)$ will be $(0,0,0)$.

4.3 Second class

Theorem 4.1. *Let β be a braid of the form $h^d\sigma_2^m$ with $m \in \mathbb{Z}$ and let $\hat{\beta}$ be the closure of β. Then, the first non-vanishing Milnor's invariant of $\hat{\beta}$ will be of weight 2.*

Proof. We can quickly calculate that

$$L(h^d) = (2d, 2d, 2d)$$

and

$$L(\sigma_2^m) = (0, m, 0)$$

Let β be a braid of the form $h^d\sigma_2^m$ with $m \in \mathbb{Z}$. Since linking number is additive, we have

$$L(\hat{\beta}) = (2d, 2d+m, 2d)$$

It follows that $L(\hat{\beta}) = (0,0,0)$ if and only if $d = 0$ and $m = 0$. Thus, the only braid β of this form whose closure has no non-zero weight 2 Milnor's invariants is the trivial braid on three strands. □

References

[Mur74] Kinio Murasugi. *On Closed 3-braids*. American Mathematical Society, 1974. ISBN: 978-0-8218-9952-6.

[Coc90] Tim D. Cochran. *Derivatives of links: Milnor's concordance invariants and Massey's products*. American Mathematical Society, 1990. ISBN: 978-0-8218-2489-4.