Exploring Solvability of the String Link Concordance Group Using Milnor's Invariants

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Introduction

How close is $C(2)/P(2)$ to being Abelian?

$C(2)/P(2)$ is the group of two-component string links quotiented by the group of two-component pure braids. In this group, we consider string links equivalent up to concordance. Recently, Kuzbary showed that $C(2)/P(2)$ is not abelian. We want to take this result further and show $C(2)/P(2)$ is not solvable. In this project, we research the Milnor's invariants of commutators in $C(2)/P(2)$; if a commutator of weight m has a non-zero Milnor's invariant, then $C(2)/P(2)$ is not m-solvable.

We use Milnor's invariants as a tool to learn about solvability of $C(2)/P(2)$. Linking number measures how difficult two components of a link are to pull apart. Milnor's invariants are a higher order version of linking number, used when linking number provides insufficient information about how components are linked.

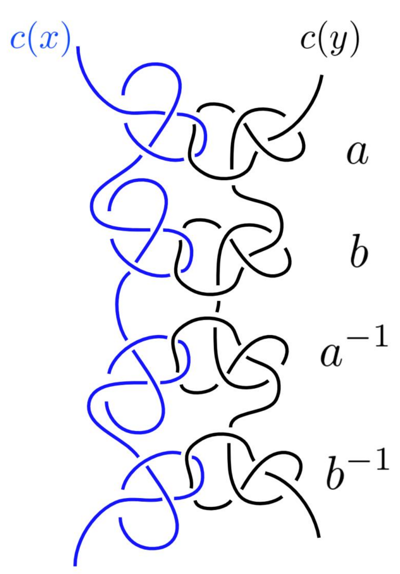
Definitions

n-COMPONENT LINK. A collection of n disjoint circles smoothly embedded in S^3.

STRING LINK. The result of splitting open a link along a disk; the closure of a string link is a link. Not every string link is a pure braid.

LINK CONCORDANCE. Two n-component links σ_1, σ_2, are concordant if there exist n smooth, embedded cylinders in $B^3 \times I$ connecting the components from each link. Concordance is weaker than isotopy but it helps define inverses in $C(2)/P(2)$. A concordance of string links can be viewed as splitting open a concordance between links.

COMMUTATOR.
$[a,b] = aba^{-1}b^{-1}$.
If $[a,b] = id_G$, then a and b commute. In this project, we work with string link commutators of various sizes from the derived series of $C(2)/P(2)$, like the one shown here, where a, b and their inverses are smaller string links stacked together.



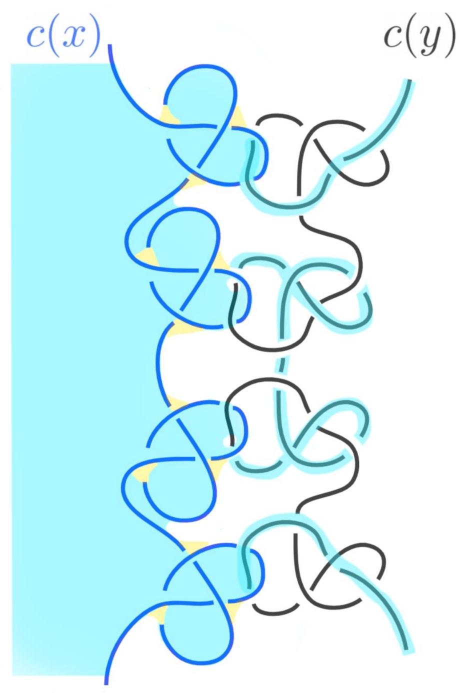
String link commutator

Methods: Computing Milnor's Invariants of a String Link Commutator

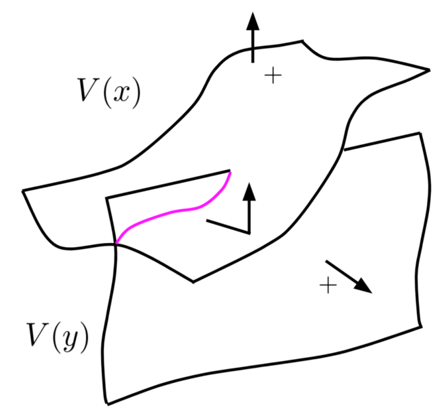
METHOD 1: Computing Milnor's Invariants using Surface Systems

SEIFERT SURFACE. A Seifert surface for a knot k is an oriented surface with k as its boundary. Seifert surfaces are not unique.

SURFACE SYSTEMS METHOD. We can generate Seifert surfaces bounded by each component of a two-component string link, then find the positive push off of their curve of intersection and generate a surface bounded by this curve. We can repeat this process and keep taking intersections of surfaces. As we generate more curves, we can take the linking numbers of each pair of curves. These linking numbers are the Milnor's invariants of the link.



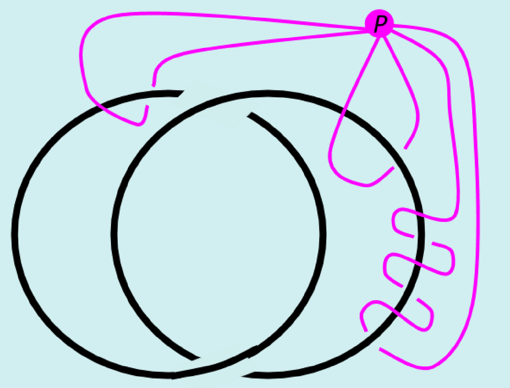
A Seifert surface for the component $c(x)$. Several hollow tubes keep the surface disjoint from $c(y)$.



Taking the positive push off of a curve of intersection by pushing the curve off the surfaces in the positive direction (dictated by both surfaces' orientation).

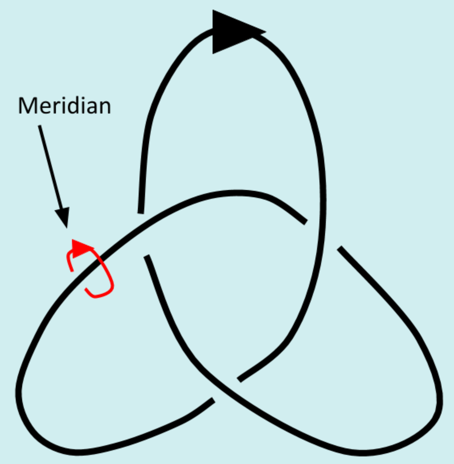
METHOD 2: Milnor's Invariants Using Fundamental Group of a Link Complement

FUNDAMENTAL GROUP. Group of loops in $S^3\backslash v(L)$ based at a shared point p, equivalent under homotopy. Written $\pi_1(S^3\backslash v(L), p)$. Group operation is concatenation at p.



A Hopf link with several elements from the fundamental group of its complement in S^3

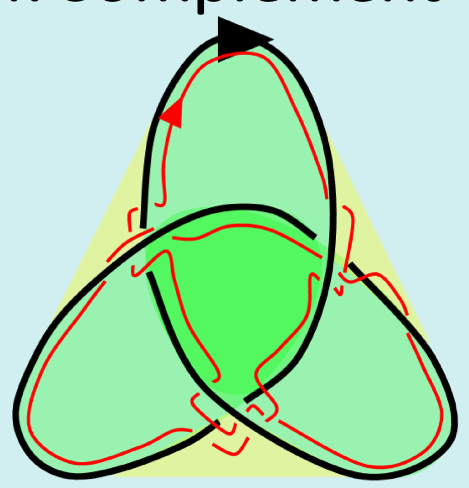
MERIDIAN. A meridian of a knot k is a loop on the boundary torus $\partial(S^3\backslash v(k))$ which bounds a disk on the interior of the torus but doesn't bound a disk on the boundary.



Trefoil with meridian (red)

LONGITUDE (BLACKBOARD). A longitude of a knot k is a loop on the boundary torus $\partial(S^3\backslash v(k))$ which intersects a meridian exactly once. A blackboard framed longitude follows the knot on one side constantly.

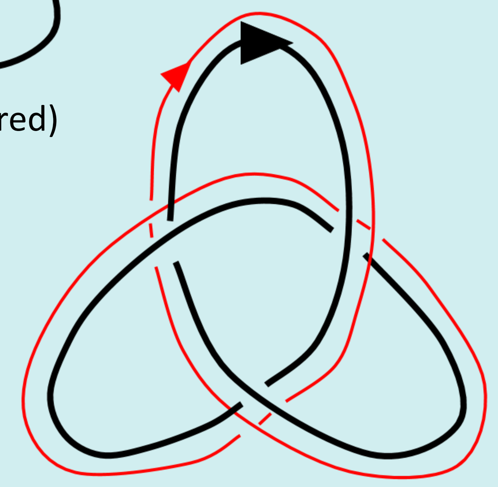
0-FRAMED LONGITUDE. The 0-framed longitude of a component k is the intersection of a Seifert surface for k with the boundary torus $\partial(S^3\backslash v(k))$. It can be used to find Milnor's invariants using Magnus expansion.



Trefoil with surface and 0-framed longitude (red)

MAGNUS EXPANSION. A mapping from the free group with n generators to the ring of power series in n non-commuting variables $X_1 \dots X_n$ where

$X_i \to 1 + X_i$
$X_i^{-1} \to 1 - X_i + X_i^2 - X_i^3 + \cdots$



Trefoil with longitude (red)

FINDING MILNOR'S INVARIANT. When the zero-framed longitude for a link component is expressed in terms of meridians and expanded with the Magnus expansion, then the coefficients of the first non-constant terms of the same degree are the first nonzero Milnor invariant for that link.

Results

THEOREM. If L is a two-component string link commutator, then the 0-framed longitude of each component is equal to its blackboard longitude.

LEMMA. For every weight $m \geq 2$ commutator c in the derived series of $C(2)/P(2)$ that is the composition of string links $a_1, a_2, \dots, a_{m+1}$ and their inverses, if $a_1 = a_3, a_2 = a_4, a_5 = a_7$, and so on for all a_n, then its closure $\tilde{c}$ is concordant to the two-component unlink and c has no non-zero Milnor's invariants.

Further Research

Is $C(2)/P(2)$ m-solvable for any weight m?

If there exists a commutator of weight m with a non-zero Milnor's invariant, we can use the tools presented and create an example of the following form:

$$\mu_{\tilde{c}}(\underbrace{1\dots1}_{m}\underbrace{2\dots2}_{m}) \neq 0$$

References

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[2] C. A. Otto, The(n)-Solvable Filtration of the Link Concordance Group and Milnor's μ-Invariants. PhD thesis, Rice University, April 2011.
[3] C. Livingston, "Knot theory," 1993.
[4] T. Cochran, "Derivatives of Links" 1990.

Acknowledgements

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