/ ON BRAID GROUPS /

by

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INTRODUCTION

Artin's Braid Group B_n on n strings can be defined as a certain group of automotphisms of a free group F_n on n free generators. Artin has shown that to every knot or linkage in 3-space there corresponds a class of conjugate elements in B_n (for some n).

For this reason, the transformation problem in B_n (i.e. the characterization of all elements of B_n which are conjugate with an arbitrarily given element) is of topological interest. One way of approaching the transformation problem is the construction of representations of B_n in terms of finite matrices, since the eigenvalues of a matrix are invariants of its class of conjugate elements.

In the present paper, we study various representations of B_n from a group theoretical point of view. We start by observing the following fact: If C is a normal divisor of F_n which admits all of the automorphisms of B_n, then B_n acts also (as a group of automorphisms) on F_n/C. This group of induced automotphisms of F_n/C is a homomorphic image of B_n. Similarly, if C admits all the automotphisms of a properly-chosen subgroup of B_n, then that subgroup acts on F_n/C, and we obtain a homomorphic image of that subgroup. If C is the second commutatot group F_n'' of F_n, we can obtain a homomorphic image of a certain subgroup I_n of B_n in this manner. Moreover, we can represent this image faithfully by matrices of order n, whose elements belong to a ring generated by n indeterminates and their inverses. If we wish to obtain a representation of the whole group B_n, we may do this in either one of two different ways. One way is to set our n indeterminates equal to a single indeterminate x, and thus to obtain a representation which was found by Burau. In doing so, we may be adding additional relations to our braid group; we can prove that Burau's matrices form a faithful representation of B_n acting on a quotient group of F_n, where the normal divisor contains F_n'' as a proper subgroup. The other possible method is to take advantage of the fact that I_n is of finite index in B_n. Thus the group B_n as it acts on F_n/F_n'' can be represented faithfully by a group of matrices which are still of finite order, although this order is higher than n. Whether these matrices represent B_n faithfully, however, is still an open question. No elements of B_n are known, other than the identity, which leave the residue classes of F_n'' in F_n invariant, but it seems a dfficult proposition to prove that none exist.

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Definitions and background information

The Braid group B , as defined by Artin [1], is generated by σ_1 , $\sigma_2, \ldots, \sigma_n$ with the defining relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_i \longleftrightarrow \sigma_k \qquad (\longleftrightarrow \text{ menas commute with})$$

any element of this group will be called a braid.

Let $d_{ik} = \sigma_i \sigma_{i+1} \cdots \sigma_k$, where $k \geqslant i$.

Artin has shown [1] that B_n is generated by σ_1 and d_{1n}.

The defining relations for this set of generators are:

$$d_{1n}^{n} = (\sigma_1 d_{1n})^{n-1}$$

$$\sigma_1 \longleftrightarrow d_{1n}^{i} \sigma_1 d_{1n}^{-1} \text{ for } 2 \leqslant i \leqslant \tfrac{n}{2}$$

The center of this group is the cyclic group generated by d_{1n}^{n} [2] . The group obtained from B_n by adding the relation $d_{1n} = 1$ will be denoted by B_n*.

This group B_n has a geometric interpretation [1,2] as the group of braids with n strings. There is a permutation of n objects naturally associated with every element of B_n; namely, if n objects, one placed at the upper end of each of the strings of the braid, travel down along their respective strings, they will arrive at the lower ends, in general, in a different order. This relation between braids and permutations can also be expressed in purely group-theoretic terms; namely, if we add to the relations defining B_n the additional relation $\sigma_1^2=1$, the resulting group is isomorphic to the symmetric group Σ_n; therefore Σ_n is a homomorphic image of B_n. The kernel of this homomorphism is the group I_n of all braids having the identity permutation [1]. The generators of I_n are the elements A_{ik}, where $A_{ik}=d_{ik}\sigma_{k-1}^2 d_{i,k-1}^{-1}$ for $i<k$ and $A_{ik}=A_{ki}$. Thus the number of generators of I_n is $\frac{n(n-1)}{2}$. The defining relations for I_n [2] are all commutativity relations. In the case of B_3, we see that $A_{13}A_{11}A_{12}=d_{13}^3$ which commutes with all elements of B_3 and therefore, in particular, with all elements of I_3. The pair of relations $A_{13}A_{12}A_{23} \leftrightarrow A_{12}$ and $A_{13}A_{12}A_{23} \leftrightarrow A_{23}$ turn out to be a complete set of defining relations for I_3. A consequence of this is that any two of the three generators of I_3 generate a free group. The defining relations for I_n in general fall into three classes. The first of these three classes of relations is the natural generalization of the defining relations for I_3; namely, if $i<j<k$, then $A_{ik}A_{ij}A_{jk}$ will commute with each of the three generators A_{ik}, A_{ij}, and A_{jk}. The second class of relations are consequence of the relation $\sigma_i \leftrightarrow \sigma_k$ for $|i-k|\geq 2$; namely, if i, j, k, and l are distinct and the pairs (i, j) and (k, l) do not separate each other, then

$A_{ij} \longrightarrow A_{kl}$. There is a third class of relations which is related to the center of B_4 in the same way that the first class was related to the center of B_3; namely, if $i<j<k<l$, then $A_{jl}A_{jk}A_{kl}A_{il}A_{ik}A_{ij}$ will commute with each of the six elements A_{ij}, A_{ik}, A_{il}, A_{jk}, A_{jl}, and A_{kl}. These three classes of relations form a complete set of defining relations for I_n.

Of course, the symmetric group Σ_n contains other subgroups besides the identity, and to each of these there corresponds a subgroup of B_n. For example, we may consider those permutations of n objects in which the first r of them are permuted arbitrarily and the remaining objects stay fixed. To this subgroup of Σ_n would correspond the subgroup of B_n generated by the generators of I_n and those of B_r.

The set of elements of elements A_{ik} where i remains fixed and k ranges through the integers 1,...,n (except k=i, which is excluded) generate a free group C_{in} such that $B_n/C_{in} \cong B_{n-1}$ [4]. Another subgroup of B_n which is known to be free is the group S_n generated by the elements $(d_{1i})^i\,(d_{i+1,n})^{-(n-i)}$, for i running from 1 to n-1. The group B_n^{*}/S_n is isomorphic to the group of mapping classes of a sphere with n points missing [5].

There is still another way in which B_n is connected with free groups; namely, B_n may be considered a group of automorphisms of a free group F_n with free generators $g_1, \ldots, g_n$. To σ_i there corresponds the following substitution:

$$\begin{array}{l} g_i \to g_{i+1} \\ g_{i+1} \to g_{i+1} g_i g_{i+1} \\ g_j \to g_j \text{ for } j \neq i,\ i+1 \end{array}$$

Similarly, to σ_i^{-1} there corresponds the following substitution:

$$\begin{array}{ll} g_i & g_i g_{i+1} g_i^{-1} \\ g_{i+1} & g_i \\ g_j \to & g_j \text{ for } j \neq i,\ i+1 \end{array}$$

An automorphism of F_n corresponds to an element of B_n if and only if it leaves the element $g_1 g_2 \ldots g_n$ invariant and replaces each of the elements $g_1, \ldots, g_n$ by a transform of one of these elements [1]. Of course, an element of I_n will replace each of the g_i by a transform of that same element.

One way of adding relations to B_n is by adding relations to F_n. For example, if F_n is abelianized, then we have $\sigma_1^2 = 1$ and therefore we get Σ_n from B_n by adding the relation that the g_i commute. If, instead of making the g_i commute, we merely make the commutators belong to the center, then we get the relation $(\sigma_1 \sigma_2^{-1})^3 = 1$. There is a still weaker relation that we can add to F_n; namely, instead of making the commutators belong to the center, we may merely make them

commute with each other and thus obtain F_n/F_n'' .
By making the commutators of F_n commute, we obtain a homomorphic image of B_n, which we may call B_n°. We do not know whether or not B_n° is isomorphic to B_n.

<u>Burau's representation and its properties</u>

To each generator σ_i of B_n, Burau [3] assigns the n×n matrix

$$M^n(\sigma_i)=\begin{array}{cl}\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ & & & 1-x & x & & \\ & & & 1 & 0 & & \\ & 0 & & & & 1 & \\ & & & & & & \ddots \;\; 1 \end{pmatrix} & \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix}\end{array}$$

It is easy to verify that the following relations are satisfied:

$$M^n(\sigma_i)M^n(\sigma_{i+1})M^n(\sigma_i)=M^n(\sigma_{i+1})M^n(\sigma_i)M^n(\sigma_{i+1})$$
$$M^n(\sigma_i)M^n(\sigma_k)=M^n(\sigma_k)M^n(\sigma_i) \quad \text{for } |i-k|\geq 2$$

Thus the group M^n generated by $M^n(\sigma_1),\ldots,M^n(\sigma_n)$ is a homomorphic image of B_n, and, if b_1 and b_2 are any two elements of B_n, we may define $M^n(b_1 b_2)=M^n(b_1)M^n(b_2)$.
Burau has also shown [3] that for any matrix in M^n, the elements m_{ik} of this matrix satisfy the following two relations:

$$m_{i1}+\ldots+m_{in}=1$$
$$m_{1i}+xm_{2i}+\ldots+x^{n-1}m_{ni}=x^{i-1}$$

These two properties make it possible for Burau's representation to be reduced. For we can consider the matrix with elements m_{ik} as defining a linear transformation of the n-dimensional space whose elements are vectors $(1, 1, \ldots, 1)$, where the transformation is given by $\xi_i' = m_{i1}\xi_1 + \cdots + m_{in}\xi_n$. We see that the vector $(1,1,\ldots 1)$ is always transformed into itself; also the hyperplane $\xi_1 + x\xi_2 + \cdots + x^{n-1}\xi_n = 0$ is always transformed into itself. The vector will lie in the hyperplane only if $1 + x + \cdots + x^{n-1} = 0$, therefore only if x is an nth root of unity different from 1 itself. In any other case, the vector will lie outside the hyperplane, and therefore a linear transformation which keeps the hyperplane pointwise fixed and also keeps the vector fixed must be the identity transformation. Therefore the transformations of the hyperplane which are induced by the transformations corresponding to matrices in M^n form a group which is isomorphic to M^n, unless x is an nth root of unity different from 1. To obtain a matrix representation of the transformations of this hyperplane, we simply eliminate ξ_1, by the substitution $\xi_1 = -x\xi_2 - x^2\xi_3 - \cdots - x^{n-1}\xi_n$. As a result, we get a linear transformation of the (n-1)-dimensional space of vectors $(\xi_2, \xi_3, \ldots \xi_n)$. The matrix of this transformation will be of order n-1. Thus we have

Theorem 1. From any matrix $M^n(x)$, we can obtain a corresponding matrix $R^n(x)$ of order n-1, such that the group R^n is isomorphic to M^n for all values of x except nth roots of unity different from 1. This result enables us to investigate the properties of M^n by working with matrices of order n-1. For all values of x except the nth roots of unity different from 1, we shall obtain exactly the same relations whether we work with M^n or R^n. If, however, we let x be one of the n-th roots of unity different from 1, we may expect relations to be satisfied in R^n which do not hold in M^n. We should, however, in this case, be interested in both the relations in M^n and those in R^n, since the two sets of relations are obtained from isomorphic groups by means of the same operation; namely, replacing the free variable x by a root of unity. The properties of R^n in this case are also interesting, since in this case, and in no other, R^n can be semi-reduced. The reason for this is not difficult to see. If we calculate the matrices of R corresponding to σ_1 and d_{1n}, we obtain

$$R^n(\sigma_1)=\begin{pmatrix} -x & -x^2 & \cdots & -x^{n-1} \\ & 1 & \ddots & 0 \\ 0 & & \ddots & 1 \end{pmatrix},\quad R^n(d_{1n})=\begin{pmatrix} -x & -x^2 & \cdots & -x^{n-1} \\ 1 & \ddots & & 0 \\ 0 & & \ddots & \\ & & 1 & 0 \end{pmatrix},\quad R^n(\sigma_i)=M^{n-1}(\sigma_{i-1})$$
for $i=2,\ldots,n$.

Hence it is clear that no linear relation among the elements of a column, such as the one already observed in M^n, can exist in R^n. As for the rows, it is apparent that the only such relation possible is the same one that exists in M^n; namely, that the sum of the elements in any row is 1. This relation will be satisfied by the first row of $R^n(\sigma_1)$, however, only if x is one of the n-th roots of unity different from 1. If x is a primitive m-th root of unity, where $m|n$, we can obtain from a matrix $R^n(u)$ a matrix $t_m R^n(u)\, t_m^{-1}$, where

$$t_m = \begin{pmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix} \qquad t_m^{-1} = \begin{pmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

In the matrix resulting from this transformation, the first column will consist entirely of zeros except for the element is the first row, which will be 1. Hence we have

<u>Theorem 2. The matrices R^n can be semi-reduced if x is one of the n-th roots of unity, and this is the only reduction possible.</u>

We shall denote by R^n_m the group of matrices obtained from R^n by substituting for x a primitive m-th root of unity, and by T^n_m the group of matrices obtained by crossing out the first row and the first column from $t_m R^n_m t_m^{-1}$. Many interesting results may be obtained by letting m assume various values, but we wish to postpone consideration of such results to a later discussion. At present we shall limit our attention to a special case by means of which we may establish the faithfulness of Burau's representation of B_3; namely, that of putting $x=-1$ in R^3_2. The two generators then become a pair of generators of the modular group, for which a complete set of relations is known [6]. This set of relations consists of the relations defining B_3 together with the additional relation $(\sigma_1\sigma_2)^6 = 1$. Since $(\sigma_1\sigma_2)^6$ commutes with all elements of B , the only elements of B_3 which are not 1 which will become 1 when the relation $(\sigma_1\sigma_2)^6 = 1$ is added are the positive and negative powers of $(\sigma_1\sigma_2)^6$, which will be represented in M^3 by matrices whose determinants are the corresponding powers of x^6. Since none of these matrices is the unit matrix, there can be no relations satisfied in M^3 which are not satisfied in B_3. Therefore we have

<u>Theorem 3. M^3 is a faithful representation of B_3.</u>.

<u>A derivation of Burau's representation and its generalization</u>

We have already considered B_m as a group of automorphisms of a free group F_m. We now wish to consider B_m as acting, not on F_m, but on F_m/F_m''. We assign to each of the g_i a 2×2 matrix, containing indeterminates x_i and a_i as follows:

$$g_i \rightarrow \begin{pmatrix} x & a_i \\ 0 & 1 \end{pmatrix}$$

This is a faithful representation of F_m/F_m''. [7,8]
Now let us consider the effect of the automorphisms of B_m on these matrices. First we replace each of the σ_i by its inverse. (It is clear, from the nature of the defining relations of B_m, that replacing each

of the σ_i by its inverse produces an automorphism of B_n.) Therefore, we let the following automorphism correspond to σ_i :

$$g_i \to g_i g_{i+1} g_i^{-1}$$
$$g_{i+1} \to g_i$$
$$g_j \to g_j \text{ for } j \neq i, i+1.$$

In terms of our matrices, we have the following transformations:

$$\begin{pmatrix} x_i & a_i \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} x_{i+1} & (1-x_{i+1})a_i + x_i a_{i+1} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_{i+1} & a_{i+1} \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} x_i & a_i \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_j & a_j \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} x_j & a_j \\ 0 & 1 \end{pmatrix} \text{ for } j \neq i, i+1.$$

We observe that the second row never changes. The first element in the first row is always one of the x_i; indeed, the x_i are permuted by any braid according to the permutation which corresponds naturally to that braid. It is the second element in the first row, however, that deserves the most attention. We observe that this element is a linear combination of the a_i where the coefficients are expressions involving the x_i. Thus, to each of the σ_i, and, more generally, to each of the elements of B_n, there corresponds a linear transformation of the a_i. The matrix of the linear transformation corresponding to the σ_i is as follows:

$$\begin{array}{cl} \left(\begin{matrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1-x_{i+1} & x_i & & \\ & & & 1 & 0 & & \\ & & & & & 1 \ddots & \\ & & & & & & 1 \end{matrix}\right) & \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix} \end{array}$$

If we dropped the subscripts, what we would have would be precisely Burau's representation. Thus we appear, not only to have derived Burau's representation, but also to have obtained a generalization of Burau's representation, which has a better chance of being faithful than does Burau's original representation. However, if we test our generalized matrices, we find that the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ is satisfied only if $x_1 = x_2 = \ldots = x_n = x$. Therefore we see that we cannot hope to represent B_n by a generalization of Burau's representation. However, our derivation gives us some insight into the structure of Burau's representation. We see that Burau's representation is a representation of a group of automorphisms of the group Q^n generated by the matrices

$$\begin{pmatrix} x & a_i \\ 0 & 1 \end{pmatrix}$$

Q^n is the group obtained from F_n by adding the relation that any two words in the g_i such that the sum of the exponents in each of them is zero commute with each other. [7,8] This relation can also be expressed in the somewhat simpler form that, for any i, j, and k, $g_i g_j^{-1} g_k = g_k g_j^{-1} g_i$.

Every automorphism in this group will

replace each of the a_i by a linear combination of the a_i, and will leave the other three elements of the matrix unchanged. Therefore the automorphism of Q^n produced by an element of B_n is completely determined by Burau's representation of that element, and thus we have

Theorem 4. Burau's representation of B_n is a faithful representation of B_n acting on Q^n.

We see that this is the best result we can obtain if we wish to represent the whole group B_n. However, there is a possibility that we may obtain a still better result if we work only with I_n since, although the defining relations for B_n imply those of I_n, the converse may not be true. Indeed, we find that the matrices in $x_1, \ldots, x_n$ corresponding to I_n satisfy all the defining relations for I_n. Furthermore, we observe that an automorphism corresponding to an element of I_n replaces each of the a_i, in the matrices representing the generators of F_n/F_n'', by a linear combination of the a_i, and leaves each of the other three elements of the matrix unchanged. In particular, each of the x_i appearing in the upper left-hand corner of the matrix reappears there as a result of the automorphism, since the elements of I_n all have the identity permutation. Therefore the generalized Burau matrix representing an element of I_n determines completely the automorphism of F_n/F_n'' produced by that element. Hence we have

<u>Theorem 5. The generalized Burau matrices, involving the n variables $x_1, \ldots, x_n$ instead of the single variable x, corresponding to the elements of I_n, form a faithful representation of I_n.</u>

<u>Some free subgroups of B_n</u>

We recall that there were several free groups contained in I_n. Therefore, if we could prove that Burau's representation was faithful, we could establish that several different groups of matrices of order n or n-1 involving a single variable x are free. If we could prove that our generalization of Burau's representation is faithful, we would, similarly, have several free groups of matrices of order n involving n variables. We should like to establish that, in the case of the generalized Burau representation, the converse of this statement is also true. This would mean that the statement that the generalized Burau's representation is free is equivalent with the statement that a certain group of matrices is free. In order to do this, we must make use of the fact that there is a certain value of n for which our representation of I_n has been proved faithful. For n=1 or 2, this faithfulness is trivial; we have also proved it for n=3. We know that the generalized Burau's representation of I_{n+1} can be gotten by adding the relation that commutators commute to F_{m+1}. Now, we may add to F_{m+1} still another relation; namely, $g_1 = 1$. If we let I_{m+1} act on this group, what we get is simply I_m. Now, if a braid was represented by the identity matrix beforethe new relation was added to the free group, it will certainly be represented by the identity after the relation is added, since putting a generator equal to 1 will not make a commutator cease to be a commutator. But since we know that our representation of I_m is faithful, we know that, if the braid we started with was not the identity, its being represented by the identity matrix in our representation of I_m is due entirely to the new relation $g_1 = 1$. Therefore our braid must be one which would become the identity if we added to F_{m+1} the relation $g_1 = 1$. But such a braid must be an element of the free group generated by $A_{11}, A_{12}, \ldots, A_{mk}$. Similarly, if instead of letting $g_1 = 1$, we had let $g_k = 1$, we would have proved that this braid must be an element of the free group generated by $A_{1k}, \ldots, A_{mk}$. Therefore we have

Theorem 6. Let b be a braid which gives the identity matrix in the generalized Burau's representation, and let k run through the values 1, 2, ...,n. Then for each of these values of k the free group generated by $A_{1k}, \ldots, A_{mk}$ must contain b.

This theorem states that the braid b must be an element of n different free groups, all of which are normal divisors of B_n. Similarly, by assigning special values to x, we obtain other normal divisors of B_n which must contain b. It appears at first sight that if we know several such subgroups containing b, it ought to be possible to prove that the only element common to all these subgroups is the identity, and thus to prove faithfulness for the generalized Burau representation, or possibly even for the original Burau representation. In the case, however, where these subgroups are normal divisors, as they are in every case that we have, unless we have an infinite number of such normal divisors, there will always be elements different from the identity which belong to all of them. To obtain such elements, we observe that if we have a normal divisor containing an element b_1 and another normal divisor containing b_2, the element $a_2 = b_1 b_2 b_1^{-1} b_2^{-1}$ will belong to both normal divisors, and unless the two normal divisors are such that any element of one commutes with any elements of the other, we can always find an a_2 different from the identity. Similarly, if we have n normal divisors, from which we choose representatives $b_1, \ldots, b_n$, we may define recursively $a_1 = b_1$, $a_i = a_{i-1} \; b_i \; a_{i-1}^{-1} \; b_i^{-1}$, and so obtain an element a_n which belongs to all our normal divisors. In the case of our free groups, if we take them consecutively, we certainly do not get the identity.

In the particular case $n=4$, we have found four free groups which must contain any element for which the generalized Burau representation gives the identity matrix. In this case, however, it turns out that such a braid must also be contained in the free group S_4. In fact, it must be contained in that subgroup of M_4 which is generated by $(\sigma_1 \sigma_2)^6$ and its transforms. To prove this statement, we must put $x=-1$. (The theorem we are proving now, unlike the previous theorem, applies to the original Burau representation as well as to its generalization.) It is clear that if a braid will vanish if we put $(\sigma_1 \sigma_2)^6 = 1$, the matrix representing that braid will become the identity. We must now establish the converse of this theorem. In doing so, it will be more convenient to work with the reduced representation than with the original Burau representation. In doing so, however, we should be aware of the fact that we may be adding relations to our group by this reduction, since $(-1)^4 = 1$. In fact, we do add the additional relation $(\sigma_1 \sigma_2 \sigma_3)^4 = 1$, as can easily be verified. But, since $(\sigma_1 \sigma_2 \sigma_3)^4$ generates the center of B_4, this is a rather harmless relation. It is easily seen that if we put $x=-1$ in the original Burau representation, $(\sigma_1 \sigma_2 \sigma_3)^{4n} \neq 1$ for any integer $n \neq 0$. It follows that if putting $x=-1$ in the reduced Burau representation adds only the two relations $(\sigma_1 \sigma_2)^6 = 1$ and $(\sigma_1 \sigma_2 \sigma_3)^4 = 1$, then putting $x=-1$ in the original Burau representation will add only the single relation $(\sigma_1 \sigma_2)^6 = 1$. Therefore our proof is complete if we can show that the only relations added to B_4 by putting $x=-1$ in the reduced Burau representation are $(\sigma_1\sigma_2)^6=1$ and $(\sigma_1\sigma_2\sigma_3)^4=1$. First, let us apply to the generators of our group R_2^4 that transformation by which T_2^4 is obtained. We get

$$\sigma_1 \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \sigma_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

We observe that the matrices representing σ_1 and σ_3 differ only in the element in the upper right-hand corner. In fact, T_2^4 is generated by σ_1 and σ_2 and is isomorphic to M_2^3. (Incidentally, an analogous relation exists in B_4 itself without the relation $(\sigma_1\sigma_2)^6=1$; namely, if we add to B_4 the relation $\sigma_1=\sigma_3$, we get a group isomorphic to B_3, as is easily seen by examining the defining relations for B_4 and B_3. Thus we see that B_3 is contained in B_4 not only as a subgroup, but also as a quotient group. For $n>4$, no such quotient group exists. This is due basically to the fact that the commutativity relation between the σ_i is no longer transitive.) We also observe that in R_2^4 the elements $\tau_1=\sigma_3^{-1}\sigma_1$ and $\tau_2=\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$ generate a free abelian normal divisor. The group R_2^4 is generated by σ_1, σ_2, τ_1, and τ_2. The word problem has been solved for the group generated by σ_1 and σ_2 [1,2]; it is trivial for the group generated by τ_1 and τ_2. If we have a set of relations by means of which we can express any element of R_2^4 as a word in σ_1 and σ_2 followed by a word in τ_1 and τ_2, we can express any element of R_2^4 in a unique form. Therefore a complete set of defining relations for R_2^4 consists of the defining relations for the group generated by σ_1 and σ_2 (including the relation ($\sigma_1 \sigma_2$ $)^6=1$), the relation $\tau_1 \longleftrightarrow \tau_2$ which is the only defining relation for the group generated by τ_1 and τ_2, and the four relations which will express $\sigma_1 \tau_1 \sigma_1^{-1}$, $\sigma_2 \tau_1 \sigma_2^{-1}$, $\sigma_1 \tau_2 \sigma_1^{-1}$, and $\sigma_2 \tau_2 \sigma_2^{-1}$ as elements of the abelian group generated by τ_1 and τ_2. We wish to show that these relations are all consequences of the defining relations for B_4 and the two additional relations $(\sigma_1 \sigma_2)^6=1$ and ($\sigma_1 \sigma_2 \sigma_3$ $)^4=1$. In fact, we get the following transformation formulas, which are consequences of the defining relations for B :

$$\sigma_1 \tau_1 \sigma_1^{-1} = \tau_1$$
$$\sigma_2 \tau_1 \sigma_2^{-1} = \tau_2$$
$$\sigma_1 \tau_2 \sigma_1^{-1} = \tau_2 \tau_1^{-1}$$
$$\sigma_2 \tau_2 \sigma_2^{-1} = \tau_2 \tau_1^{-1} \tau_2$$

Finally, from the relation $\tau_1 \longleftrightarrow \tau_2$, we get ,
$\sigma_1 \sigma_2 \sigma_3^2 \sigma_2 \sigma_1 = (\sigma_2 \sigma_3)^3$. But ($\sigma_2 \sigma_3$ $)^6=1$ is a consequence of ($\sigma_1 \sigma_2$ $)^6=1$, therefore our relation becomes $\sigma_1 \sigma_2 \sigma_3^2 \sigma_2 \sigma_1 (\sigma_2 \sigma_3)^3 = 1$. But $\sigma_1 \sigma_2 \sigma_3^2 \sigma_2 \sigma_1 (\sigma_2 \sigma_3)^3 = (\sigma_1 \sigma_2 \sigma_3)^4$. This completes the proof of [illegible]

<u>Theorem 7. The group R_2^4 is isomorphic to the group obtained from B_4</u>

by adding the two relations $(\sigma_1\sigma_2)^6 = 1$ and $(\sigma_1\sigma_2\sigma_3)^4 = 1$. The group M_2^4 is isomorphic to the group obtained from B_4 by adding the single relation $(\sigma_1\sigma_2)^6 = 1$. The normal divisor of B_4 generated by $(\sigma_1\sigma_2)^6$ and its transforms, which is a free group with an infinite number of free generators and is contained in S_4, must contain any element of B_4 for which Burau's representation gives the identity matrix.

In the proof of Theorem 7, we observed that the formulas expressing the transforms of τ_1 and τ_2 by σ_1 and σ_2 as elements of the group generated by τ_1 and τ_2 depended only upon the defining relations for B_4. Therefore τ_1 and τ_2 are the generators of the normal divisor of B_4 consisting of all elements which would become 1 if we put $\sigma_1 = \sigma_3$; When we found that, in R_2^4, τ_1 and τ_2 generate a free abelian group, however, we made essential use of the relations $(\sigma_1\sigma_2)^6 = 1$ and $(\sigma_1\sigma_2\sigma_3)^4 = 1$. We should now like to find what relations are satisfied by τ_1 and τ_2 in B_4, without these additional relations. We wish to show that the commutator subgroup of the group generated by τ_1 and τ_2 is contained in S_4. In the first place, the permutations corresponding to τ_1 and τ_2 commute, and therefore all our commutators will have the identity permutation. Now, S_4 can be defined geometrically as the group of braids with identity permutation that can be produced if the ends of the four strings are attached to a sphere, by merely rotating the sphere successively about various axes perpendicular to the direction of the strings of the braid and letting the sphere pass between the strings [5]. It is clear geometrically, if we draw the braids τ_1 and τ_2, that each of these elements can be produced by such a rotation, although it does not give the identity permutation If, however, we consider an element generated by τ_1 and τ_2 and belonging to the commutator subgroup (or, indeed, any element generated by τ_1 and τ_2 having identity permutation), it will satisfy all the conditions required for an element to belong to S_4. Since S_4 is a free group, and any subgroup of a free group is free, it follows that the commutator subgroup of the group generated by τ_1 and τ_2 is free. Thus any element of the group generated by τ_1 and τ_2 can be expressed uniquely in the form $\tau_1^{m_1}\tau_2^{m_2}c$, where c is an element ofthe commutator group which is determined by the given element. But this means that τ_1 and τ_2 generate a free group. Unlike all other free groups that we have found contained in braid groups, this one has generators whose permutations are not identity. We have obtained a new solution to the word problem in

B_4, since any element in B_4 can be expressed uniquely as a product of an element of B_3 and an element of the free group generated by τ_1 and τ_2. Also, we are now able to represent elements of B_3 by automorphisms of a free group having only two free generators, instead of one having three generators, as originally defined.

Theorem 8. The subgroup of B_4 generated by τ_1 and τ_2 is free, and the elements of B_3 may be represented as automorphisms of this free group as follows:

$$\sigma_1 : \begin{array}{l} \tau_1 \to \tau_2 \\ \tau_2 \to \tau_2 \tau_1^{-1} \end{array}$$

$$\sigma_2 : \begin{array}{l} \tau_1 \to \tau_2 \\ \tau_2 \to \tau_2 \tau_1^{-1} \tau_2 \end{array}$$

We observe that the element $c_0 = \tau_1^{-1} \tau_2 \tau_1 \tau_2^{-1}$ is invariant under these automorphisms. This is reflected in the fact that $(\sigma_1 \sigma_2)^6$, which replaces σ_1 and σ_2 by $c_0^{-1}\sigma_1 c_0$ and $c_0^{-1}\sigma_2 c_0$ respectively, commutes with all elements of our group of automorphisms. Therefore, if we abelianize the free group on which our automorphisms act, we add the relation $(\sigma_1 \sigma_2)^6 = 1$ to our group of automorphisms. But if we abelianize the free group, then we may write ourtransformations additively as follows:

$$\sigma_1: \quad \tau_2' = \tau_2 - \tau_1$$
$$\tau_1' = \tau_1$$

$$\sigma_2: \quad \tau_2' = 2\tau_2 - \tau_1$$
$$\tau_1' = \tau_2$$

We observe that the matrices of these linear transformations are precisely those defined by R_2^3, and therefore that if we let τ_1 and τ_2 commute, σ_1 and σ_2 generate a group isomorphic to R_2^3 whose defining relations are $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ and $(\sigma_1 \sigma_2)^6 = 1$. In order to establish by an argument similar to that of Theorem 3 that if τ_1 and τ_2 become free generators, σ_1 and σ_2 generate a group isomorphic to B_3, it is only necessary to show that if τ_1 and τ_2 are free generators, $(\sigma_1 \sigma_2)^6$ is an element of infinite order. But clearly transforming by c_0 is an operation of infinite order. Therefore we have

Theorem 9. B_3 is isomorphic to a group of automorphisms of a free group with two free generators τ_1 and τ_2. The generators of this group of automorphisms are

$$\sigma_1: \quad \tau_1 \to \tau_1$$
$$\tau_2 \to \tau_2 \tau_1^{-1}$$

$$\sigma_2: \quad \tau_1 \to \tau_2$$
$$\tau_2 \to \tau_2 \tau_1^{-1} \tau_2$$

Extension of the generalized Burau representation

We have already seen the original Burau representation of B_n acting on a group in which elements in which the sum of the exponents of the generators is zero commute, and we have also seen the generalized Burau representation which represents faithfully a group of automorphisms of the group obtained by adding to a free group the relation that commutators commute, but which applies not to B_n but only to I_n. We should like to combine the advantages of these two representations, by obtaining a representation of the whole group B_n which faithfully represents B_n acting on the group obtained from the free group by making commutators commute. It turns out that this can indeed be done, although at the expense of increasing the rank of the representation from n to n·n!

Thus, for example, if the generalized Burau representation of I_4 by 4x4 matrices is faithful, then whether Burau's original representation of B_4 is faithful or not, we can construct a group of 96x96 matrices which we can easily prove is faithful. These can, in fact, be considered 24x24 matrices whose elements are not numbers but elements of I_4. If we set all elements of I_4 equal to the identity in our matrices, we obtain a representation of the group obtained from B_4 by putting the elements of I_4 equal to the identity. But this is simply the symmetric group Σ_4, of order 24. But we do not even have to deal with 24×24 matrices, for we can carry out the extension process in several steps. We recall that there is a subgroup of B_4 corresponding to every subgroup of Σ_4 (or, more generally, there is a subgroup of B_n corresponding to every subgroup of Σ_n.) Thus, if we have a representation of I_n, we can extend it first to a representation of the group generated by σ_1 and the generators of I_n. This will increase the rank of our representation from n to 2n, but the new matrices may be considered to be 2×2 matrices whose elements are elements of I_n. If we have a representation of the group generated by the generators of I_n and those of B_m, we can obtain a representation of the group generated by the generators of I_n and those of B_{m+1}. In doing so, we increase the rank from $n \cdot m!$ to $n \cdot (m+1)!$, but the new matrices may be considered to be $(m+1)\times(m+1)$. Matrices whose elements are elements of the group generated by the generators of I_n and those of B_m. Thus, given a representation of I_n, we can proceed recursively to build up a representation of B_n without, at any particular stage of the process, having to deal with matrices of degree $>n$. The set of defining relations for the resulting representation will consist of the defining relations for B_n and the defining relations for the given representation of I_n. In order to make this possible, what we need is a method for finding a faithful representation of a group, containing a subgroup of finite index by means of matrices of degree equal to that index whose elements are elements of the subgroup. But such a method is provided by the following procedure [9]. First, we choose a representative of each of the right cosets of our subgroup. In order to find a matrix representing a given generator of our group, we must first find the corresponding permutation matrix. If we multiply each of our representatives on the right by a particular generator, a new set of elements will be produced, no two of which will belong to the same coset. Thus we obtain a new set of representatives of the cosets, but, in general, in a different order unless the generator with which we started was already an element of our subgroup. There will be some permutation of the cosets thus obtained which will produce the original cosets. Thus to every generator of our group, there corresponds a permutation matrix, which is the identity if our generator is an element of the subgroup. To obtain a faithful representation of our group, we simply replace each 1 in our permutation matrix by a properly chosen element of the subgroup. To obtain such an element, we again multiply each representative on the right by our generator. We obtain a representative of some coset, which may or may not be the same as the representative of that coset which we have chosen. In any case, we multiply our result on the right by the inverse of the representative of that coset which we have chosen. Thus, we may get the identity if the two representatives turn out to be the same; otherwise we obtain some other element of our subgroup. By means of this process, we obtain a faithful representation of our group [9]. Applying this procedure to our braid group, either directly, using I_n as the subgroup and B_n as the whole group, or recursively in the manner that we have indicated, we obtain

Theorem 10 Given a faithful representation of I_n acting on a certain group, we can derive from it a faithful representation of B_n acting on that group. In particular, we can obtain a faithful representation of B_n acting on F_n/F_n''.

We may illustrate the process just discussed by finding a faithful representation of B_3, given a faithful representation of I_3. As representatives we use 1, σ_1, σ_2, $\sigma_1\sigma_2$, $\sigma_2\sigma_1$, and $\sigma_1\sigma_2\sigma_1$. As generators of B_3 we use σ_1 and σ_2. (We also might use Artin's two generators σ_1 and d_{13}. In the case of B_3 there is no advantage in doing this, but for n>3, we may, by using Artin's pair of generators, reduce the number of generators for which we must find representations from n-1 to 2.) By following the procedure we have just outlined, we obtain

$$\sigma_1 \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \sigma_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \sigma_2\sigma_1^2\sigma_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2^2 & 0 & 0 \end{pmatrix} \qquad \sigma_2 \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sigma_2^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_1\sigma_2^2\sigma_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sigma_1^2 & 0 \end{pmatrix}$$

It is easily verified that these matrices satisfy the relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$. Furthermore if we interpret the braids that appear as elements of these matrices to be the matrices corresponding to those braids in the generalized Burau representation, we know, in this case, that we have a faithful representation of B_3, since even the original Burau representation of B_3 is faithful. Thus the more elaborate representation of B_3 that we have obtained is no improvement over Burau's original representation. But it is possible to obtain a representation of B_n for $n>3$ exactly the same method that we have just used to obtain a representation of B_3. In this case, we know neither, on the one hand, whether our more elaborate representation is any improvement over Burau's original representation, nor, on the other hand, whether the improvements, if any, which we have made are sufficient to make the resulting group of matrices a faithful representation of B_n.

A consideration of special values of x.

We have devoted much attention to Burau's representation, its properties, and its generalizations. But we may also consider what happens to Burau's representation if we substitute particular values for x. The resulting groups of matrices will be homomorphic images of M_n. We may also condider what happens to our 2x2 matrices representing the free group on which B_n acts, if we substitute particular values for the x's. The resulting group will, in general, no longer be free, and its automotphisms will form a group which is a homomorphic image of B_n and which can be represented by Burau's matrices if we substitute for x the same value which we substituted into the 2x2 matrices representing our free group. As for possible values of x, the only one that we cannot use is 0, since then our matrices would become degenerate. If we put x=1, our free group becomes abelian, and our braid group becomes the symmetric group Σ_n. If we let x be a complex number, we should choose a root of unity if we wish to add as many relations as possible to our groups, since otherwise a word will be equal to 1 only if the sum of the exponents of the generators in that word is zero. As before, we shall indicate that we have substituted a primitive nth root of unity for x by attaching the subscript m to the symbol in question; thus, we may speak of $M_m^{\sim}$, $R_m^{\sim}$, $T_m^{\sim}$, or $Q_m^{\sim}$. Also, we note that x does not necessarily have to be a complex number; it could, for example, be any non-singular matrix. We may attempt either, to find all the relations which are satisfied if we give x a particular value or to determine all values of x for which a particular relation is satisfied. The first of these two problems, of course, is the more significant, but the second is easier to solve, since it amounts merely to solving a system of simultaneous equations in a single unknown. We may, for instance, wish to find those roots of unity for which certain elements of our group are of finite order. Naturally, we should wish to know the order of σ_1, for this will also be equal to the order of all the σ_i. Another element whose order we should like to know is $\sigma_1 \sigma_2$. This will also give us the order of all elements of the form $\sigma_i \sigma_{i+1}$ or $\sigma_i \sigma_{i-1}$. In particular, we are especially interested in the order of $(\sigma_1 \sigma_2)^3$, since this is the center of B_3. Still another element of special importance is $\sigma_1 \sigma_2^{-1}$. With the knowledge of the orders of these three elements $\sigma_1, \sigma_1\sigma_2$, and $\sigma_1 \sigma_2^{-1}$, we know the order of any word of length 2 in the σ_i. But, aside from this the element $\sigma_1 \sigma_2^{-1}$ possesses special properties which make it worthy of consideration. In the first place, if we put $\sigma_1 \sigma_2^{-1} = 1$, B_n becomes a cyclic group.

Then, $(\sigma_1 \sigma_2^{-1})^3 = \sigma_1^2 \sigma_2^2 \sigma_1^{-2} \sigma_2^{-2}$; Thus, the order of $\sigma_1 \sigma_2^{-1}$ tells us something about the order of the commutator of two generators of I_n. Geometrically, $(\sigma_1 \sigma_2^{-1})^3$ represents a linkage of three circles in which no two of them are linked. The matrices representing powers of $(\sigma_1 \sigma_2^{-1})$ also have a special property; namely, they are functions of $x+\frac{1}{x}$ and therefore are real if x is an imaginary root of unity. All these considerations seem to justify our imaginary selection of σ_1, $\sigma_1 \sigma_2$, and $\sigma_1 \sigma_2^{-1}$ as the three elements of B_n in whose orders we are most interested. Since these are all elements of B_3, we may determine their orders by considering their representation in R^3_m. The results obtained are as follows:

<u>Theorem 11. The n-th power of $R^3_m(\sigma_1)$ is 1 if and only if $\frac{1-(-x)^n}{1+x} = 0$ where x is a primitive m-th root of unity. $R^3_m(\sigma_1 \sigma_2)$ is of order 3m unless m is divisible by 3, in which case $R^3_m(\sigma_1 \sigma_2)$ is of order m. If m=2 or 3, $R^3_m(\sigma_1 \sigma_2^{-1})$ is of infinite order. For m=1,4, and 6, the order of $R^3_m(\sigma_1 \sigma_2^{-1})$ is 3,6, and 4 respectively.</u>

From the above results it appears that the values of m for which the most relations are satisfied are 1, 4, and 6. For m=1, we aready know that we get the symmetric group. For m=4 and m=6, it will also turn out that R^3_m is a finite group. In the case of m=4, the defining relations are

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$
$$\sigma_1^4 = 1$$
$$(\sigma_1 \sigma_2)^{12} = 1$$
$$(\sigma_1 \sigma_2^{-1})^6 = 1$$

These relations are sufficient to define a group of order 96, as can be seen by drawing the graph of the group. To show that no further defining relations are necessary, we might calculate our 96 matrices to make sure they are all distinct. We may, however, simplify our procedure somewhat by observing that the $R^3_4((\sigma_1 \sigma_2)^3)$ is simply -i times the unit matrix. If we put $(\sigma_1 \sigma_2)^3=1$, then R^3_4 becomes isomorphic to Σ_4. If we can list 24 matrices which are all distinct, no two of which can be obtained from each other by multiplication by i,1, or -i, this will be suffieient, and we do not have to exhibit 96 matrices. Our 24 matrices are as follows:

$$R^3_4(\sigma_1) = \begin{pmatrix} -i & -i \\ 0 & 1 \end{pmatrix}$$

$$R^3_4(\sigma_2) = \begin{pmatrix} -i & 0 \\ 1 & 1 \end{pmatrix}$$

$$R^3_4(\sigma_1^2) = \begin{pmatrix} -1 & -1-i \\ 0 & 1 \end{pmatrix}$$

$$R^3_4(\sigma_2^2) = \begin{pmatrix} i & -1 \\ 0 & 1 \end{pmatrix}$$

$$R^3_4(\sigma_1^3) = \begin{pmatrix} i & -1 \\ 0 & 1 \end{pmatrix}$$

$$R^3_4(\sigma_2^3) = \begin{pmatrix} i & 0 \\ -i & 1 \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2) = \begin{pmatrix} -1-i & -i \\ 1 & 1 \end{pmatrix}$$

$$R^3_4(\sigma_2\sigma_1) = \begin{pmatrix} -1 & -1 \\ -i & 1-i \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2\sigma_1\sigma_2) = \begin{pmatrix} i & -1 \\ -i & 1-i \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2\sigma_1) = \begin{pmatrix} -1+i & -1 \\ -i & 1-i \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2\sigma_1^3) = \begin{pmatrix} 1+i & i \\ -1 & -i \end{pmatrix}$$

$$R^3_4(\sigma_2\sigma_1^2) = \begin{pmatrix} i & i-1 \\ -1 & -i \end{pmatrix}$$

$$R^3_4(\sigma_2\sigma_1^3) = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$$

$$R^3_4(\sigma_2\sigma_1^3\sigma_2) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$$

$$R^3_4(\sigma_1^3\sigma_2) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$R^3_4(\sigma_2^3\sigma_1) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2\sigma_1^3) = \begin{pmatrix} 1-i & 1 \\ i & 0 \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2\sigma_1\sigma_2^3) = \begin{pmatrix} 0 & i \\ -1 & -1-i \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2^3) = \begin{pmatrix} 0 & -i \\ -i & 1 \end{pmatrix}$$

$$R^3_4(\sigma_1\sigma_2^2) = \begin{pmatrix} -1 & -i \\ 1-i & 1 \end{pmatrix}$$

$$R^3_4(\sigma_2^2\sigma_1) = \begin{pmatrix} i & i \\ -i-1 & -i \end{pmatrix}$$

$$R^3_4(\sigma_2^2\sigma_1^2) = \begin{pmatrix} 1 & 1+i \\ i-1 & -1 \end{pmatrix}$$

$$R^3_4(\sigma_1^2\sigma_2) = \begin{pmatrix} -1 & -1-i \\ 1 & 1 \end{pmatrix}$$

$$R^3_4(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These matrices are indeed distinct, and no two of them are such that one can be obtained from the other by multiplication by i, -1, or -i. Therefore we have

Theorem 11. R^3_4 is a group whose exact order is 96, which is generated by σ_1 and σ_2 with the following defining relations:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$
$$\sigma_1^4 = 1$$
$$(\sigma_1\sigma_2)^{12} = 1$$
$$(\sigma_1\sigma_2^{-1})^6 = 1$$

Each of the above 24 matrices represents a coset of B_3. We have assigned to each matrix a particular element of B_3, but, of course, we could have done this in many ways. We have chosen a particular set of representatives which will satisfy a Schreier condition, in order that we may use the Reidemeister-Schreier method to find a complete set of defining relations for the group of elements which become 1 when we add to B_3 the relations defining R'_4 and the additional relation $(\sigma_1 \sigma_2)^3 = 1$. Upon application of the Reidemeister-Schreier method, we obtain

Theorem 12. The subgroup of B_3 which becomes 1 if we add the relations defining R and the additional relation $(\sigma_1 \sigma_2)^3 = 1$ is generated by the following six elements: $\sigma_1^4, \sigma_2^4, (\sigma_1 \sigma_2^{-1})^3, (\sigma_2^{-1}\sigma_1)^3, (\sigma_2 \sigma_1^{-1} \sigma_2)^2$, and $(\sigma_1 \sigma_2)^3$. The first five of these generate a free group, and the sixth belongs to the center.

The significance of this theorem is that it tells us the generators and relations for a group which will contain any element of B_3 for which the corresponding matrix is the unit matrix. Incidentally, it was not necessary to add the relation $(\sigma_1 \sigma_2)^3 = 1$. By adding this relation, however, we are able to reduce considerably the amount of labor involved in the Reidemeister-Schreier method, since we have only 24 cosets to work with instead of 96.

Turning to B_4 we obtain the following result:

Theorem 13. The group T_4^4 consists of 96 matrices which are identical with those of R_4^3. The elements of R_4^4 which correspond to the identity in T_4^4 form an abelian group with two generators, each of infinite order.

Just as we have been able to obtain such results by putting $m=4$, we ought to be able to get eqally good results for $m=6$, since there, too, $\sigma_1 \sigma_2^{-1}$ is of finite order. The relations given by Theorem 10 are:

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$
$$\sigma_1^3 = 1$$
$$(\sigma_1 \sigma_2)^6 = 1$$
$$(\sigma_1 \sigma_2^{-1})^4 = 1$$

By drawing the graph of this group, we see that the order of the group is 24, and that $(\sigma_1 \sigma_2)^6 = 1$ is a consequence of the other three relations. This group, however, cannot be isomorphic to Σ_4, since $\sigma_1 \sigma_2$ is of order 6 while the order of an element of Σ_3 must be either 2, 3, or 4. Thus we have

Theorem 14. R_6^3 is a group of order 24, not isomorphic to Σ_4. Its generators are σ_1 and σ_2, and the defining relations are:

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$
$$\sigma_1^3 = 1$$
$$(\sigma_1 \sigma_2^{-1})^4 = 1$$

Let us see what $\sigma_1^3 = 1$ means in terms of F_n. The automorphism corresponding to σ_1^3 is

$$g_1 \to g_1 g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1^{-1}$$

$$g_2 \to g_1 g_2 g_1 g_2^{-1} g_1^{-1}$$

Thus, if we add to B_n the relation $\sigma_1^3 = 1$, we are adding to F_n the relation $g_1 g_2 g_1 = g_2 g_1 g_2$. Thus we have exactly the same relation among the g's that we have among the σ 's in B_3, although the relation $(\sigma_1 \sigma_2^{-1})^4 = 1$ will impose an additional rather complicated relation on the g's. Thus, in R_6^3, we have, in a sense, a braid group acting on a braid group.

SUMMARY

We have constructed and investigated the properties of several matrix representations of braid groups, and characterized these representations in terms of group theory. Some interesting results have been obtained from Burau's representation M^n by putting the parameter x equal to various roots of unity. We have attempted, unsuccessfully, to decide whether M^n is a faithful representation of B_n. We have, however, been able to find a generalized Burau representation which lies between B_n and M^n, so that now we have

$$B_n \to \text{generalized Burau representation} \to M^n.$$

We do not know whether either of these two homomorphisms is an isomorphism.

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