SOME ASPECTS OF CLASSICAL KNOT THEORY *

by

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0. Introduction

Man's fascination with knots has a long history, but they do not appear to have been considered from the mathematical point of view until the 19th century. Even then, the unavailability of appropriate methods meant that initial progress was, in a sense, slow, and at the beginning of the present century rigorous proofs had still not appeared. The arrival of algebraic-topological methods soon changed this, however, and the subject is now a highly-developed one, drawing on both algebra and geometry, and providing an opportunity for interplay between them.

The aim of the present article is to survey some topics in this theory of knotted circles in the 3-sphere. Completeness has not been attempted, nor is it necessarily the case that the topics chosen for discussion and the results mentioned are those that the author considers the most important: non-mathematical factors also contributed to the form of the article.

For additional information on knot theory we would recommend the survey article of Fox [43], and the books of Neuwirth [112] and Rolfsen [128]. Reidemeister's book [125] is also still of interest. As far as problems are concerned, see [44], [112], [113], [75], as well as the present volume. Again, we have by no means tried to include a complete bibliography, although we hope that credit for ideas has been given where it is due. For a more extensive list of early references, see [26].

In the absence of evidence to the contrary, we shall be working in the smooth category (probably), and homology will be with integer coefficients.

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1. Enumeration

It seems that the first mathematician to consider knots was Gauss, whose interest in them began at an early age [31, p. 222]. Unfortunately, he himself wrote little on the subject [49, V, p. 605; VIII, pp. 271-286], despite the fact that he regarded the analysis of knotting and linking as one of the central tasks of the 'geometria situs' foreseen by Leibniz [49, V, p. 605]. His student Listing, however, devoted a considerable part of his monograph [88] to knots, and in particular made some attempt to describe a notation for knot diagrams.

A more successful attack, inspired by Lord Kelvin's theory of vortex atoms, was launched in the 1860's(1) by the Scottish physicist Tait. His first papers on knots were published in 1876-77 (see [145]). Later, with the help of the 'polyhedral diagrams' of the Reverend Kirkman, Tait and Little (the latter had done some earlier work [90]) made considerable progress on the enumeration ('census') problem, so that by 1900 there were in existence tables of prime knots up to 10 crossings and alternating prime knots of 11 crossings [91], [92], [93], [145].

Essentially nothing was done by way of extending these tables until about 1960, when Conway invented a new and more efficient notation which enabled him to list all (prime) knots up to 11 crossings and all links up to 10 crossings [19], (revealing, in particular, some omissions in the 19th century tables).

There are two main aspects of this kind of enumeration: completeness and non-redundancy. One wants to know (i.e. prove) that one has listed all knots up to a given crossing number, and also that the knots listed are distinct. The former belongs to combinatorial mathematics, and although a proof of completeness throughout the range of the existing tables would no doubt be long and tedious, it is not hard to envisage how such a proof would go. Indeed, implicit in the compilation of the tables is the possession of at least the outline of such a proof. Although some omissions in Conway's tables have recently been brought to

(1) see Maxwell's letter of 1867 quoted in [77, p. 106]

crossing number	3	4	5	6	7	8	9	10
number of prime knots	1	1	2	3	7	21	49	165

(See [3] for pictures of knots up to 9 crossings, and [115] for those with 10 crossings.) There are 550 11-crossing knots now known [117], and although there is a good chance that these might be all, the task of proving them distinct is a formidable one that has not yet been completed. Indeed, as intimated in [117] (which contains some partial results), invariants more delicate than those which suffice up to 10 crossings are now required.

2. The Group

The knot problem becomes discretized when looked at from the point of view of combinatorial topology. It is noted in [30], for example, that it can be formulated entirely in terms of arithmetic. However, this kind of 'reduction' seems to be of no practical value, nor does it seem to have any theoretical consequences (for decidability, for example). There are also many natural numerical invariants of a knot which may be defined, such as the minimal number of crossing points in any projection of the knot, the minimal number of crossing-point changes required to unknot the knot (the 'gordian number' [160]), the maximal euler characteristic of a spanning surface (orientable or not), and so on (see [125, pp. 16-17]). But these tend to be hard to compute.

The first successful algebraic invariant to be attached to a knot was the fundamental group of its complement, (the group of the knot), and presentations of certain knot groups appear fairly early in the literature (see [146]). General methods for writing down a presentation of the knot group from a knot projection were given by Wirtinger (unpublished (?) ;see [125, III, §9])and Dehn [27]. Actually it was soon recognized [28] that a knot contains (at least a priori) more information than just its group, as we now explain. Let $K \subset S^3$ be our given (smooth) knot, and let X be its exterior, that is, the closure of the complement of a tubular neighbourhood N of K. (The exterior and the complement are equivalent invariants: clearly the exterior determines the complement, and the

Despite such examples, the group is still a powerful invariant. It was shown by Dehn [27], for example, (modulo his 'lemma', which was introduced specifically for this purpose) that the only knot with group $\mathbb{Z}$ is the unknot. This finally became a theorem in 1956 when Dehn's lemma was established by Papakyriakopoulos [114]. At the same time, Papakyriakopoulos also proved the first version of the sphere theorem, and as a consequence, the asphericity of knots, that is, the fact that the complement of a knot is a $K(\pi,1)$. It follows that the group of a knot determines the homotopy type of its complement.

The role of the peripheral structure was finally completely clarified by Waldhausen's work [155] on irreducible, sufficiently large, 3-manifolds (this work in turn being based on earlier ideas of Haken). Specializing to the case that concerns us here, Waldhausen showed that if K_1 and K_2 are knots with exteriors X_1, X_2, then any homotopy equivalence of pairs $(X_1,\partial X_1) \to (X_2,\partial X_2)$ is homotopic to a homeomorphism. This implies, for example, that knots (under the strongest form of equivalence, which takes both the ambient orientation and that of the knot into account), are classified by (isomorphism classes of) their associated triples $(\pi K,\lambda,\mu)$. We may remark that it is a purely algebraic exercise to pass from such a classifying triple to a classifying group [20]. Other, more complicated, but more geometric, ways of nailing down the peripheral structure within a single group are given in [140], [163] and [37]. (The classifying groups obtained there are, respectively, the free product of the groups of two cables about $K \# K_0$ (where K_0 is, say, the figure eight knot), the group of the double of K, and the group of the (p,q)-cable of K where $|p| \geq 3$ and $|q| \geq 2$.)

The situation may to some extent be summarized by the following diagram, where, for simplicity, $\sim$ now denotes the weak form of knot equivalence which disregards orientations, (and P_i denotes the peripheral subgroup $\pi_1(\partial X_i)$).

$$\begin{array}{ccccccccc}
K_1 \sim K_2 \Rightarrow X_1 \cong X_2 & \Leftrightarrow & (X_1,\partial X_1) \simeq (X_2,\partial X_2) & \Rightarrow & X_1 \simeq X_2 \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
(\pi K_1,\lambda_1,\mu_1) \cong (\pi K_2,\lambda_2^{\pm 1},\mu_2^{\pm 1}) & \Rightarrow & (\pi K_1,P_1) \cong (\pi K_2,P_2) & \Rightarrow & \pi K_1 \cong \pi K_2
\end{array}$$

even a proof of a negative answer to either (1) or (2)), seems beyond the scope of existing techniques. Question (3) may be easier. (Indeed it follows from the finiteness theorem of Thurston mentioned above that if K is a hyperbolic knot, and some non-trivial resewing of a tubular neighbourhood of K gives S^3, then the new knot is at least not isotopic to K. For, if the resewing in question is the one which 'kills' $\mu\lambda^n$, say, $n \neq 0$, then the new knot's being isotopic to K would imply the existence of a self-homeomorphism h of the exterior X of K taking $\lambda \mapsto \lambda^\varepsilon$, $\mu \mapsto \mu^\varepsilon \lambda^{\varepsilon n}$, where $\varepsilon = \pm 1$. Since h^r would then take $\mu \mapsto (\mu\lambda^{rn})^{\pm 1}$, the resewing corresponding to $\mu\lambda^{rn}$ would yield S^3, for all r, contradicting the finiteness statement.)

Returning to our diagram of implications, the example of the reef and granny shows that the horizontal implications on the right are not reversible. On the other hand, Johannson [66], [67] and Feustel [36] have shown that if $\pi K_1 \cong \pi K_2$, and X_1 contains no essential annuli, then $X_1 \cong X_2$. Now the only knots whose exteriors contain essential annuli are composite knots and cable knots. The cable knots with unknotted core are just the torus knots, and they are known to be determined by their group [14]. So let K be a non-trivial knot, and let $K_{p,q}$ denote the (p,q)-cable about K, that is, a curve on the boundary ∂N of a tubular neighbourhood N of K, homologous in ∂N to $p[m] + q[\ell]$. (Here, p and q are coprime integers with $|q| \geq 2$, and (ℓ, m) is a longitude-meridian pair on ∂N.) Feustel-Whitten [37] have shown that if $|p| \geq 3$, then $\pi K_{p,q}$ determines $K_{p,q}$. So prime knot complements are known to be determined by their group except possibly for cable knots $K_{p,q}$ with $|p| \leq 2$.

The problem concerning these remaining cable knots turns out to be related to the general question of whether knots are determined by their complement. More precisely, suppose there exist inequivalent knots K_1, K_2 with homeomorphic exteriors X_1, X_2. The homeomorphism $X_1 \to X_2$ must take m_1 to a curve homologous in ∂X_2 to $\pm [m_2] + n[\ell_2]$, for some $n \neq 0$. Then Hempel (unpublished) and Simon [141] show that if there is such a counterexample, with $|n| \neq 1, 2$, or 4, then there exist cable knots of type $(\pm 1, \pm n/2)$ (n even), or $(\pm 2, \pm n)$ (n odd) with isomorphic groups, whose complements are not homeomorphic.

(1) the R-module structure of $H_1(X_k;R)$

(2) the module structure of $H_1(X_k;R)$ over the group ring $R[C_k]$.

If R is an integral domain, and $Q(\ )$ denotes field of fractions, we also have

(3) for $k < \infty$, the product structure given by the linking pairing $T_1(X_k;R) \times T_1(X_k;R) \to Q(R)/R$ on the R-torsion subgroup of $H_1(X_k;R)$. ($R = \mathbb{Z}$ is really the only case of interest here.)

(4) the product structure given by the Blanchfield pairing (see §7) $H_1(X_\infty;R) \times H_1(X_\infty;R) \to Q(R[C_\infty])/R[C_\infty]$.

We may remark here that, for $k < \infty$, it is traditional to work with the corresponding branched cyclic covering M_k, rather than with the unbranched covering X_k. Since M_k is a closed 3-manifold, and for other reasons too (see §5), this is perhaps more natural. However, the two are essentially equivalent from the present point of view, as it is not hard to show that $H_1(X_k;R) \cong H_1(M_k;R) \oplus R$, as $R[C_k]$-modules, the module structure on R being induced by the trivial action of C_k.

Apart from the obvious relationships between the above considerations (1)-(4), we have that the $R[C_\infty]$-module $H_1(X_\infty;R)$ determines the $R[C_k]$-module $H_1(X_k;R)$, $1 \leq k < \infty$, (see §5), and the Blanchfield pairing on $H_1(X_\infty;R)$ determines the linking pairing on $T_1(X_k;R)$, $1 \leq k < \infty$.

4. The Infinite Cyclic Cover

Let us first consider the $R[C_\infty]$-module $H_1(X_\infty;R)$. If t denotes the canonical multiplicative generator of C_∞, (determined by the orientations of S^3 and K), we may identify $R[C_\infty]$ with the Laurent polynomial ring $\Pi = R[t,t^{-1}]$. Since R is Noetherian, Π is also, by the Hilbert basis theorem. Furthermore, since X is a finite complex, the chain modules $C_q(X_\infty;R)$ are finitely-generated (free) Π-modules, and hence $H_1(X_\infty;R)$ is a finitely-generated Π-module.

The following argument of Milnor [96] establishes the crucial property that $t-1: H_1(X_\infty;R) \to H_1(X_\infty;R)$ is surjective. (Since $H_1(X_\infty;R)$ is finitely-generated and Π is Noetherian, it follows that $t-1$ is also injective.) The short exact

Taking $R = \mathbb{Q}$ in particular, we have a complete description of the $\Gamma = \mathbb{Q}[t, t^{-1}]$-module $H_1(X_\infty; \mathbb{Q})$ by a sequence of non-zero ideals $(\gamma_1) \subset (\gamma_2) \subset \ldots \subset (\gamma_n)$. The picture over $\Lambda = \mathbb{Z}[t, t^{-1}]$ is not quite so clear, as Λ is not a principal ideal domain, but one can define some invariants. Thus there are the elementary ideals $E_1 \subset E_2 \subset \ldots$, where E_i is defined to be the ideal in Λ generated by the determinants of all the $(n-i+1) \times (n-i+1)$ submatrices of any $m \times n$ presentation matrix for the module [164, pp. 117-121]. (We may suppose $m \geq n$ without loss of generality, and we put $E_i = \Lambda$ if $i > n$.) Even these are fairly intractable, but since Λ is a unique factorization domain, each E_i is contained in a unique minimal principal ideal (Δ_i). One thus obtains a sequence of elements $\Delta_1, \Delta_2, \ldots, \Delta_n$ of Λ, each determined up to multiplication by a unit (the only units of Λ are $\pm t^r$, $r \in \mathbb{Z}$), such that $\Delta_{i+1} | \Delta_i$, $1 \leq i < n$. Suitably normalized, Δ_i is called the i^{th} Alexander polynomial of the knot, $\Delta_1 = \Delta$ being called simply the Alexander polynomial. Equivalently, one can consider the elements λ_i defined by $\lambda_i = \Delta_i / \Delta_{i+1}$; λ_i is the i^{th} Alexander invariant. These definitions are essentially contained in Alexander's paper [2].

The surjectivity of $t-1\colon H_1(X_\infty) \to H_1(X_\infty)$ can be expressed by saying that, regarding $\mathbb{Z}$ as a Λ-module via the augmentation homomorphism $\varepsilon\colon \Lambda \to \mathbb{Z}$, $H_1(X_\infty) \otimes_\Lambda \mathbb{Z} = 0$. It follows that $\varepsilon(E_i) = \mathbb{Z}$, and hence $\varepsilon(\Delta_i) = \Delta_i(1) = \pm 1$. It seems most natural (see §8) to normalize Δ_i so that it is a polynomial in t such that $\Delta_i(0) \neq 0$ and $\Delta_i(1) = 1$. From this it is not too hard to show that if the elements γ_i of Γ which describe the direct sum decomposition of $H_1(X_\infty; \mathbb{Q})$ are normalized so as to be polynomials with integer coefficients with g.c.d. 1, such that $\gamma_i(0) \neq 0$ and $\gamma_i(1) > 0$, then $\lambda_i = \gamma_i$, $1 \leq i \leq n$. It thus transpires that in the presence of the integral information $H_1(X) \cong \mathbb{Z}$, the Alexander polynomials are essentially rational invariants.

In view of the last remark, it is no surprise that the Alexander polynomials do not in general determine the elementary ideals. For example, the knot 9_{46} in the Alexander-Briggs table and the stevedore's knot (6_1) have modules $H_1(X_\infty)$ which are, respectively, $\Lambda/(2-t) \oplus \Lambda/(2t-1)$ and $\Lambda/(2-5t+2t^2)$. In both cases, $H_1(X_\infty; \mathbb{Q})$ is the cyclic Γ-module $\Gamma/(2-5t+2t^2)$. However, for the stevedore's

If $x_1, \ldots, x_n$ generate $H_1(X_\infty)$ as a Λ-module, then $\{t^j x_i : 1 \leq i \leq n, -\infty < j < \infty\}$ generate $H_1(X_\infty)$ over $\mathbb{Z}$. Since $\Delta x_i = 0$, $1 \leq i \leq n$, we see that if the constant coefficient of Δ (and hence, by the symmetry of Δ (see §7), the leading coefficient also) is ± 1, then $H_1(X_\infty)$ is finitely-generated over $\mathbb{Z}$, and is therefore free abelian of rank $\deg \Delta$. The converse is also true. For these and other results on the abelian group structure of $H_1(X_\infty)$, see [24], (also [121]).

5. The Finite Cyclic Covers

To relate $H_1(X_k;R)$ to $H_1(X_\infty;R)$, consider the short exact sequence of chain complexes

$$0 \to C_*(X_\infty;R) \xrightarrow{t^k-1} C_*(X_\infty;R) \to C_*(X_k;R) \to 0 .$$

As before, this gives rise to an exact sequence

$$H_1(X_\infty;R) \xrightarrow{t^k-1} H_1(X_\infty;R) \to H_1(X_k;R) \to R \to 0 .$$

If we give R the trivial Π-action, and $H_1(X_k;R)$ the Π-module structure induced by the canonical covering translation, this is an exact sequence of Π-modules. From this and the fact that $H_1(X_k;R) \cong H_1(M_k;R) \oplus R$ (with the trivial Π-action on R), it follows that, as Π- or $R[C_k]$-modules,

$$H_1(M_k;R) \cong \operatorname{coker}(t^k-1)^{(2)} .$$

This relation between $H_1(M_k;R)$ and $H_1(X_\infty;R)$ can be conveniently expressed in matrix terms. Let $B(t)$ be any presentation matrix for $H_1(X_\infty;R)$ over Π, with respect to generators $x_1, \ldots, x_n$, say. Then $\operatorname{coker}(t^k-1)$ is generated

(2) Throughout this section, it is understood that this refers to the action on $H_1(X_\infty;R)$.

where ℓ_i is the number of distinct roots of λ_i which are k^{th} roots of 1. This result was first obtained by Goeritz [52], by explicitly diagonalizing $B(T)$ over $\mathbb{C}$.

Note that (as was pointed out in [52]), $H_1(M_k;\mathbb{C})$, or equivalently, the first Betti number of M_k, does not just depend on the Alexander polynomial $\Delta = \lambda_1, \ldots, \lambda_n$. The order of $H_1(M_k)$, however, does. Indeed, using Goeritz's diagonalization it may be shown that

$$\text{order } H_1(M_k) = |\det B(T)| = \left|\prod_{i=1}^{k} \Delta(\omega^i)\right|, \quad \text{where} \quad \omega = e^{\frac{2\pi i}{k}} .$$

(This was first observed by Fox [41]; the proof given there, however, needs some modification.)

The behaviour of $H_1(M_k)$ as a function of k is sometimes quite interesting. For example, if k is odd, then $H_1(M_k)$ is always of the form $G \oplus G$ [119], [54]. Other results, in particular, necessary and sufficient conditions for $H_1(M_k)$ to be periodic in k, are given in [55].

We shall mention Seifert's work on branched cyclic covers [136], [137] in §8.

6. The Group Again

Let π be the group of a knot K. Since covering spaces of the exterior X of K correspond to subgroups of π, much of the material discussed in §§3-5 can be expressed in purely group-theoretic terms. Thus $\pi_1(X_\infty)$ is just the commutator subgroup π' of π, so $H_1(X_\infty)$ is isomorphic to π'/π''. The Λ-module structure of $H_1(X_\infty)$ can also be described group-theoretically: let $z \in \pi$ be any element which maps to the chosen generator t of C_∞; then the action of t on $H_1(X_\infty)$ corresponds to conjugation by z on π'/π''. Hence, given some presentation of π, it will be possible to derive a Λ-module presentation for π'/π''. If the presentation of π is in turn obtained in some way from a projection of K, we will then have a recipe for computing the Λ-module π'/π'' from a knot diagram. The algorithms described by Alexander [2] and Reidemeister [125, II, §14] are of this kind, based respectively on the Dehn and Wirtinger presentations of the knot group.

unique $\mathbb{Z}$-linear function

$$\frac{\partial}{\partial x_j} : \mathbb{Z}[F] \to \mathbb{Z}[F]$$

such that

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

and
$$\frac{\partial (uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j} .$$

If w is any word in the x_j's, regarded as a loop in X based at p, w lifts to a unique path $\tilde{w}$ starting at $\tilde{p}$. It may then be readily verified (for example, by induction on the length of w) that, as a 1-chain in $\tilde{X}$,

$$\tilde{w} = \sum_{j=1}^{n} \alpha\varphi\left(\frac{\partial w}{\partial x_j}\right)\tilde{x}_j .$$

In particular, with respect to the $\mathbb{Z}[H]$-bases $\{\tilde{D}_i: 1 \le i \le m\}$, $\{\tilde{x}_j: 1 \le j \le n\}$, $\partial_2: C_2(\tilde{X}) \to C_1(\tilde{X})$ is given by the $m \times n$ matrix

$$\left(\alpha\varphi\left(\frac{\partial r_i}{\partial x_j}\right)\right) .$$

One also sees that $\partial_1: C_1(\tilde{X}) \to C_0(\tilde{X})$ is given by

$$\partial_1(\tilde{x}_j) = (\alpha\varphi(x_j) - 1)\tilde{p} .$$

The short exact sequence

$$0 \to \ker \partial_1 \to C_1(\tilde{X}) \to \operatorname{im} \partial_1 \to 0$$

gives, after factoring out by $\operatorname{im} \partial_2$, the short exact sequence (of $\mathbb{Z}[H]$-modules)

$$\overline{H_q(X_\infty,\partial X_\infty;R)} \cong H^{3-q}(\mathrm{Hom}_\Pi(C'_*,\Pi)) \ ,$$

where — denotes the conjugate module in which the action of $\pi \in \Pi$ is defined by $a \longmapsto \bar{\pi}a$. We are mainly interested in the case $q = 1$. Let us then note that since $H_1(\partial X_\infty;R)$ is generated by the boundary of the lift of a Seifert surface, $H_1(\partial X_\infty;R) \to H_1(X_\infty;R)$ is zero, and hence $H_1(X_\infty;R) \cong H_1(X_\infty,\partial X_\infty;R)$.

Now suppose R is a field, so that Π is a principal ideal domain. Then, by the universal coefficient theorem and the fact that $H_2(X_\infty;R)$ is Π-torsion, (the surjectivity of $t-1$ on $H_2(X_\infty;R)$ follows in the same way as for $H_1(X_\infty;R)$), we get

$$\overline{H_1(X_\infty;R)} \cong \mathrm{Ext}_\Pi(H_1(X_\infty;R),\Pi) \ .$$

Since $H_1(X_\infty;R)$ is also Π-torsion, we finally obtain the fundamental duality isomorphism

$$\overline{H_1(X_\infty;R)} \cong H_1(X_\infty;R) \ .$$

In particular, taking $R = \mathbb{Q}$, this implies the familiar duality property of the Alexander polynomials

$$(\Delta_i) = (\bar{\Delta}_i) \ , \quad \text{i.e.} \quad \Delta_i(t) = t^{\deg \Delta_i}\,\Delta_i(t^{-1}) \ .$$

(Note that this, and the fact that $\Delta_i(1) = 1$, implies that $\deg \Delta_i$ is even.)

Now consider the case $R = \mathbb{Z}$. Levine [85] shows that, since Λ has global dimension 2, the universal coefficient spectral sequence still gives us an isomorphism

$$\overline{H_1(X_\infty)} \cong \mathrm{Ext}_\Lambda(H_1(X_\infty),\Lambda) \ .$$

It follows from this, incidentally, that $H_1(X_\infty)$ is $\mathbb{Z}$-torsion-free. (Here is the

Consider the case when R is a field. Since the adjoint to β_R is the composition

$$\overline{H_1(X_\infty;R)} \cong \mathrm{Ext}_\Pi(H_1(X_\infty;R),\Pi) \cong \mathrm{Hom}_\Pi(H_1(X_\infty;R),Q(\Pi)/\Pi) ,$$

β_R is non-singular. Here, the first isomorphism comes from duality and universal coefficients, and the second from the short exact sequence

$$0 \longrightarrow \Pi \longrightarrow Q(\Pi) \longrightarrow Q(\Pi)/\Pi \longrightarrow 0 ,$$

using the fact that $H_1(X_\infty;R)$ is Π-torsion.

It turns out that this is also true when $R = \mathbb{Z}$ (see [12], [85]), that is, $\beta = \beta_{\mathbb{Z}}$ induces an isomorphism

$$\overline{H_1(X_\infty)} \cong \mathrm{Hom}_\Lambda(H_1(X_\infty),Q(\Lambda)/\Lambda) .$$

even though integral module is not self-dual, the Blanchfield form induces an isomorphism but note so — $A_0 \not\cong A_0^\#$ but $A_0 \cong \overline{A_0}$ ≅ $\overline{\mathrm{Hom}_\Lambda(A_0, \frac{Q(\Lambda)}{\Lambda})}$

As regards the classification of Blanchfield pairings, the case $R = \mathbb{Q}$ has been done, as follows. In [152], Trotter defines a function $\chi\colon Q(\Gamma)/\Gamma \longrightarrow \mathbb{Q}$ such that

$$\chi\beta_{\mathbb{Q}} : H_1(X_\infty;\mathbb{Q}) \times H_1(X_\infty;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

is non-singular, skew-symmetric, has $t\colon H_1(X_\infty;\mathbb{Q}) \longrightarrow H_1(X_\infty;\mathbb{Q})$ as an isometry, and has the property that the isomorphism class of the pair $(\chi\beta_{\mathbb{Q}}, t)$ determines the isometry class of $\beta_{\mathbb{Q}}$. But pairs consisting of a non-singular ε-symmetric ($\varepsilon = \pm 1$) bilinear form on a finite dimensional R-vector space, together with an isometry, have been classified (see [98]). The rational Blanchfield pairings are thereby also classified.

When $R = \mathbb{Z}$, a complete classification has not yet been achieved. (See [84], [152], for partial results; also §9.)

$$\alpha\colon H_1(F)\times H_1(F)\longrightarrow \mathbb{Z}$$

defined by

$$\alpha([w],[z]) = \mathrm{lk}(w^+,z) ,$$

where w,z are 1-cycles in F, w^+ is the cycle obtained by translating w off F in the positive normal direction, and lk denotes linking number in S^3. This form has the property that $\alpha-\alpha^T$ (^{T} denotes transpose) is just the intersection form on $H_1(F)$. In particular, $\det(\alpha-\alpha^T)=1$. Choosing some basis for $H_1(F)$, we get a $2h\times 2h$ <u>Seifert matrix</u> A representing α, where $h=$ genus F.

A Mayer-Vietoris argument on $X_\infty = \bigcup_{i=-\infty}^{\infty} Y_i$ shows [80] that $H_1(X_\infty)$ is presented as a Λ-module by the matrix $t\Lambda - \Lambda^T$. In particular, up to a unit of Λ, the Alexander polynomial $\Delta=\det(t\Lambda-\Lambda^T)$. Since putting $t=1$ gives the unimodular matrix $A-A^T$ (3), the properties $\varepsilon(E_i)=\mathbb{Z}$, $\varepsilon(\Delta_i)=1$, of the elementary ideals and Alexander polynomials are immediate.

The consequences of duality are also easily seen in this setting. For example the conjugate module $\overline{H_1(X_\infty)}$ is presented by the matrix $t^{-1}A-A^T$, which is equivalent to $(tA-A^T)^T$. In particular, $\overline{E}_i=E_i$ and $(\overline{\Delta}_i)=(\Delta_i)$ for all i. Since $\det(t\Lambda-\Lambda^T)\neq 0$, the presentation of $H_1(X_\infty)$ corresponding to $t\Lambda-A^T$ is actually a short free resolution

$$0\longrightarrow F_1\xrightarrow{\ \varphi\ } F_0\longrightarrow H_1(X_\infty)\longrightarrow 0 ,$$

where F_0, F_1 are free Λ-modules of rank $2h$. Hence $\mathrm{Ext}_\Lambda(H_1(X_\infty),\Lambda)\cong$ coker(Hom(φ, id)), and the latter is clearly presented by $(tA-A^T)^T$. So we derive our previous duality statement

(3) This is why it is natural, at least for $i=1$, to normalize so that $\Delta_i(1)=1$.

$$A \longmapsto \begin{bmatrix} & & & * & 0 \\ & A & & \vdots & \vdots \\ & & & * & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & & & 0 & 0 \\ & A & & \vdots & \vdots \\ & & & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

(The $*$'s record the way the handle links F.) The equivalence relation on Seifert matrices generated by congruence and these enlargements is known as S-equivalence. It will also be convenient to call two knots S-equivalent if they have S-equivalent Seifert matrices.

S-equivalence was first introduced, in an algebraic setting, by Trotter [150]. It also appears in [107]. The following remarks show that it is likely to be an important concept. Firstly, any two Seifert matrices for a given knot K are S-equivalent.(4) (Here is an outline of a proof. Let the matrices be associated with Seifert surfaces F_0, F_1 for K, and let these in turn correspond, via transversality, to maps $p_0, p_1 \colon X \longrightarrow S^1$, such that $p_0|\partial X = p_1|\partial X$, where X is the exterior of K. Then p_0, p_1 extend to $p\colon X \times I \longrightarrow S^1$, with $p_t|\partial X = p_0|\partial X$ for all $t \in I$. Transverse regularity gives a connected, orientable 3-manifold $M \subset X \times I$ such that $\partial M = F_0 \cup \partial F_i \times I \cup F_1$. Now choose a handle decomposition of M on F_0 with only 1- and 2-handles, such that the former precede the latter, and such that, regarding M as $F_0 \cup \text{collar} \cup \text{handle} \cup \text{collar} \ldots$, each handle is embedded in a level $X \times \{t\}$, and the collars are compatible with the I factor (see [72]). Then in a level between the 1- and 2-handles, M intersects X in a Seifert surface for K which is obtained from each of F_0, F_1 by adding hollow handles.) Secondly, given a Seifert matrix A for K, it is easy to see that any matrix obtained from A by a sequence of enlargements (and congruences) is also a Seifert matrix for K. (But this is not necessarily true for reductions.)

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(4) In [107], it is noted that by examining the effects of the Reidemeister moves on a knot diagram, the S-equivalence class of the Murasugi matrix can be shown to to be an invariant of K.

(i) $\Delta(1) = 1$, and

(ii) $\Delta(t) = t^{\deg \Delta} \Delta(t^{-1})$.

To do this, Seifert actually shows that any integral matrix A such that $A - A^T = J$ can be realized as a Seifert matrix. This is done by taking an orientable surface of the appropriate genus, regarded as a disc with bands, and embedding it in S^3 by twisting and linking the bands so as to realize A as the matrix (with respect to the basis of $H_1(F)$ represented by the cores of the bands) of the Seifert form. It follows (by changing basis) that any matrix A with $\det(A-A^T) = 1$ is a Seifert matrix.

It turns out that in Seifert's realization of the polynomial, the module which arises, i.e. the module presented by $tA - A^T$, is actually the cyclic Λ-module $\Lambda/(\Delta)$. By taking connected sums, it follows that any sequence of polynomials $\lambda_1, \ldots, \lambda_n$ satisfying the (necessary) conditions

(i) $\lambda_i(1) = 1$, $1 \leq i \leq n$

(ii) $\lambda_i(t) = t^{\deg \lambda_i} \lambda_i(t^{-1})$, $1 \leq i \leq n$, and

(iii) $\lambda_{i+1} | \lambda_i$, $1 \leq i < n$

can occur as the Alexander invariants of a knot. (This can be equivalently expressed in terms of the Alexander polynomials.) A different proof is given in [79].

In particular, the Γ-modules which can occur as $H_1(X_\infty;Q)$ for some knot are completely and simply characterized.

Over the integers, we have the following realization result of Levine [85], which brings in the Blanchfield pairing:

Let H be a finitely-generated Λ-module such that $t-1: H \to H$ is surjective, and let $\beta: H \times H \to Q(\Lambda)/\Lambda$ be a non-singular, sesquilinear, Hermitian pairing. Then β is the Blanchfield pairing of some knot.

To prove this, it is sufficient to show that every such β is given by $(1-t)(tA-A^T)^{-1}$ for some integral matrix A with $\det(A-A^T) = 1$. (Is there a direct algebraic proof of this?) This Levine does by showing that β may be is invariant under any of the 3 so-called Reidemeister moves [3], [123] on a knot diagram, and is therefore an invariant of the knot K. In particular, the absolute value of the determinant, and the Minkowski units C_p for odd primes p, are invariants of K, (but C_2 and the signature are not) [51].

In [137], Seifert relates C to the 2-fold branched cover M_2 of K, by observing that the latter can be obtained by cutting S^3 along the spanning surface for K corresponding to the shaded regions of the knot projection and gluing together two copies of the resulting manifold in an appropriate fashion. In particular, he shows that G is a presentation matrix for $H_1(M_2)$, and that the linking form $H_1(M_2) \times H_1(M_2) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is given by $\pm G^{-1}$, the sign depending on the orientation of M_2. (See also §12.) Note that $|\det G| = \text{order } H_1(M_2) = |\Delta(-1)|$ is always odd.

Such linking forms are classified by certain ranks and quadratic characters corresponding to each p-primary component (p an odd prime). See [135], [62]. In [120] (see also [78]) it is shown that these invariants determine the Minkowski units C_p, and, more generally, Kneser-Puppe in [76] show that in fact the linking form completely determines the equivalence class (in the above sense) of the quadratic form.

More recently, Trotter [150] considered the quadratic form given by $A + A^T$, where A is a Seifert matrix for K. (See also [107], which studies $M + M^T$, where M is the Murasugi matrix.) S-equivalence on A induces the equivalence relation on $A + A^T$ generated by congruence and addition of a hyperbolic plane $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This is a stronger equivalence than the one discussed previously. Also, it may be shown that if the shaded surface F obtained from a knot projection happens to be orientable, then the corresponding Goeritz matrix coincides with $A + A^T$ for some Seifert matrix A associated with F. Finally, for any Seifert matrix A of K, $A + A^T$ is in the equivalence class of Goeritz matrices of K. This may be seen by isotoping the given Seifert surface, regarded as a disc with bands, so that the bands cross over as shown in Figure 2, where + denotes one side of the surface and - the other. The modification shown in Figure 2 produces an orientable surface obtainable from the indicated knot projection by shading; the

$$H^q(X_\infty, \partial X_\infty; R) \times H^{2-q}(X_\infty, \partial X_\infty; R) \to H^2(X_\infty, \partial X_\infty; R) \cong R$$

is non-singular. Taking $q = 1$ and setting

$$\langle x, y\rangle = x \cup (ty) + y \cup (tx)$$

then defines a non-singular, R-valued, symmetric bilinear form $\langle\, ,\, \rangle$ on $H^1(X_\infty, \partial X_\infty; R)$. With respect to an appropriate basis, $\langle\, ,\, \rangle$ is given by $A + A^T$, where A is a non-singular Seifert matrix, and thus coincides with Trotter's quadratic form (tensored with R). (See [34] for details.)

We remark that the non-singularity of the above cup product pairing can be interpreted as a Poincaré duality in X_∞ of formal dimension 2. However, this non-singularity definitely fails over $\mathbb{Z}$; for example, $H^1(X_\infty, \partial X_\infty)$ $(\cong H^1(X_\infty))$ is often zero.

Taking $R = \mathbb{R}$, let λ be a symmetric, irreducible factor of the Alexander polynomial, so $\lambda = (t-\xi)(t-\bar{\xi})$ where $\xi = e^{i\theta}$, say. Milnor [96] then defines $\sigma_\theta(K)$ to be the signature of the restriction of $\langle\, ,\, \rangle$ to the λ-primary component. The signature of the knot $\sigma(K)$ is the sum of all the $\sigma_\theta(K)$.

These signatures $\sigma_\theta(K)$ turn out to be equivalent to the signature function σ_K; Matumoto has shown [94] that $\sigma_\theta(K)$ is just the jump in σ_K at $e^{i\theta}$.

12. Some 4-Dimensional Aspects

It is enlightening to consider the branched cyclic covers from a 4-dimensional point of view. The basic construction is the following. Pushing the interior of a Seifert surface F for K in S^3 into the interior of the 4-ball B^4 gives a properly embedded surface $\hat{F} \subset B^4$ with $\partial\hat{F} = K$. For $1 \le k < \infty$, we then have $M_k = \partial V_k$, where M_k, V_k is the k-fold branched cyclic cover of (S^3, K), $(B^4, \hat{F})$ respectively.

Let us first consider the case $k = 2$. In S^3, choose a thickening $F \times [-1,1]$ of $F \times 0$. Then V_2 may be constructed by taking two copies of B^4, and identifying (x,t) in one copy with $(x,-t)$ in the other, for all $x \in F$, $t \in [-1,1]$, as an orthogonal direct sum $E_0 \oplus E_1 \oplus \ldots \oplus E_{k-1}$, where E_r is the ω^r-eigenspace of τ. Let $\sigma_r(V_k)$ be the signature of the restriction of our Hermitian form to E_r. It then turns out that

$$\sigma_r(V_k) = \mathrm{sign}((1-\omega^{-r})A + (1-\omega^r)A^T), \qquad 0 \leq r < k$$

where A is a Seifert matrix for F. (See [32], [154], [18]). These signatures $\sigma_r(V_k) = \sigma_K(\omega^r)$, $0 < r < k$, are the <u>k-signatures</u> of the knot K. In particular, $\sigma_1(V_2)$ is just the signature of V_2.

We saw earlier that $\sigma_K(\xi)$ depends only on K. Here, rather more is true. We could construct V_k with $\partial V_k = M_k$ using any (orientable) surface $F \subset B^4$ with $\partial(B^4, F) = (S^3, K)$. Then $\sigma_r(V_k)$ is independent of F. To see this, we shall use the G-signature theorem [6]; (for an elementary proof for semi-free actions in dimension 4, which is all that is needed here, see [57]). Recall that the τ^s-signatures $\mathrm{sign}(\tau^s, V_k)$ are defined as follows. We have $H_2(V_k; \mathbb{C}) = H^+ \oplus H^- \oplus H^0$, where the Hermitian 'intersection' form is $\pm$-definite on $H^{\pm}$ and zero on H^0. Then

$$\mathrm{sign}(\tau^s, V_k) = \mathrm{trace}(\tau^s | H^+) - \mathrm{trace}(\tau^s | H^-) \; .$$

By similarly decomposing each eigenspace $E_r = E_r^+ \oplus E_r^- \oplus E_r^0$, we may take $H^+ = E_0^+ \oplus E_1^+ \oplus \ldots \oplus E_{k-1}^+$, etc., which (recalling that $\sigma_0(V_k) = 0$) shows that

$$\mathrm{sign}(\tau^s, V_k) = \sum_{r=1}^{k-1} \omega^{rs} \sigma_r(V_k) \; , \qquad 0 < s < k \; .$$

Inverting, we obtain

$$\sigma_r(V_k) = \frac{1}{k} \sum_{s=1}^{k-1} (\omega^{-rs} - 1)\,\mathrm{sign}(\tau^s, V_k) \; , \qquad 0 < r < k \; .$$

some $\lambda \in \Lambda$. This is enough to show that C_1 is not finitely-generated. Later, the concordance invariance of the signature was proved [107]. (Murasugi works entirely with his matrix M, but as we remarked earlier, this is a particular Seifert matrix.) This implies the existence of elements of infinite order in C_1. The Miwkowski units are also concordance invariants [106], as are the p-signatures [148] and the signatures $\sigma_\theta(K)$ [96].

This information is all subsumed under the invariance of the 'Witt class' of the Seifert form, which we shall discuss soon, but we pause briefly to consider the signature function $\sigma_K\colon S^1 \longrightarrow \mathbb{Z}$ of §11, as a direct approach to this is possible via branched covering spaces.

Recall (§12) that for $\xi = e^{\frac{2\pi ri}{k}}$ a k^{th} root of 1, $\sigma_K(\xi) = \sigma_r(V_k)$, the signature of the restriction to the ξ-eigenspace of the intersection form on the k-fold branched cyclic cover V_k. Now suppose $(S^3 \times I, T)$ is a concordance, between knots K_0 and K_1, say, and let W_k be its k-fold branched cyclic cover. If k is a prime-power, then (as in §5), $H_*(W_k;\mathbb{Q}) \cong H_*(S^3 \times I;\mathbb{Q})$; in particular, $H_2(W_k;\mathbb{Q}) = 0$. Hence, by Novikov additivity of the eigenspace signatures, $\sigma_{K_0}(\xi) = \sigma_{K_1}(\xi)$. (In particular, the p-signatures of Tristram are concordance invariants.) Since the roots of 1 of prime-power order are certainly dense in S^1, and since σ_K is continuous except at finitely many points in S^1, it follows that $\sigma_{K_0} = \sigma_{K_1}$ almost everywhere. Hence if we define $\tau_K\colon S^1 \longrightarrow \mathbb{Z}$ by taking the average of the one-sided limits of σ_K at each point, we see that τ_K is a concordance invariant. This is equivalent to the concordance invariance of the $\sigma_\theta(K)$'s, proved in [96] (see §11). Compare also [81, p. 242]. (Note that τ_K takes values in $\mathbb{Z}$, since if ξ is not a root of the Alexander polynomial of K, $A(\xi)$ (see §11) is non-singular; hence $\sigma_K(\xi) \equiv \operatorname{rank} A(\xi) \pmod 2$ is even. Also, Matumoto has shown [94] that if the 1st Alexander invariant (or minimal polynomial) λ_1 has no repeated roots, then $\tau_K = \sigma_K$.)

We now turn to the Seifert form. (The treatment which follows is that of Levine [81], [82].) Let K be a slice knot, so $(S^3,K) = \partial(B^4,D)$ for some smooth 2-disc D. A tubular neighbourhood of D in B^4 may be identified with $D \times D^2$; let $V = B^4 - D \times \operatorname{int} D^2$ be the exterior of D. Then $\partial V = X \cup D \times S^1$, where X is isometry t, such that ± 1 is not an eigenvalue of t. The classes of such <u>isometric structures</u>, under the equivalence relation obtained by factoring out forms with a t-invariant subspace of half the total dimension on which $\langle\, ,\, \rangle$ vanishes, form a Witt group $W_0(C_\infty, \mathbb{Q})$ under $\oplus$. An isomorphism

$$W_S(\mathbb{Q}) \longrightarrow W_0(C_\infty, \mathbb{Q})$$

is induced by sending a non-singular matrix A representing a class in $W_S(\mathbb{Q})$ to the class in $W(C_\infty, \mathbb{Q})$ with matrix representatives $(A + A^T, A^{-1}A^T)$. (Every class in $W_S(\mathbb{Q})$ has a non-singular representative.) Note that if A is a non-singular Seifert matrix for a knot K, then $A + A^T$ represents the quadratic form of K, and $A^{-1}A^T$ represents the automorphism $t\colon H_1(X_\infty; \mathbb{Q}) \longrightarrow H_1(X_\infty; \mathbb{Q})$.

A complete set of invariants for $W_0(C_\infty, \mathbb{Q})$ has been given by Levine [82], using results of Milnor [98]. These are defined for each λ-primary component V_λ, where λ is a symmetric, irreducible factor of the characteristic polynomial of t, and are: the exponent mod 2 of λ in the characteristic polynomial, the signature of the restriction of $\langle\, ,\, \rangle$ to V_λ (this is the σ_θ of §11), and a Witt class invariant version (analogous to a Minkowski unit) of the Hasse invariant of the restriction of $\langle\, ,\, \rangle$ to V_λ. In particular, $W_0(C_\infty, \mathbb{Q}) \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/4)^\infty \oplus (\mathbb{Z}/2)^\infty$. The image of the injection $W_S(\mathbb{Z}) \longrightarrow W_0(C_\infty, \mathbb{Q})$ is also isomorphic to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/4)^\infty \oplus (\mathbb{Z}/2)^\infty$.

A different but related approach to the computation of $W_S(\mathbb{Z})$ is described by Kervaire in [73]. For further results on the structure of $W_S(\mathbb{Z})$, see [143].

Similar definitions and results hold for knots of S^{4n+1} in S^{4n+3} for $n > 0$; in particular, there is a knot concordance group C_{4n+1} and a homomorphism

$$\psi_{4n+1}\colon C_{4n+1} \longrightarrow W_S(\mathbb{Z}) .$$

Levine has shown that, if $n > 0$, ψ_{4n+1} is an isomorphism [81]. According to Casson-Gordon [17], [18], however, this is not the case for $n = 0$. We shall briefly summarize their argument.

It can be shown that $\tau(K,\chi)$ is independent of r and V_k.

Now suppose that K is a slice knot, so $(S^3,K) = \partial(B^4,D)$, say. Let W_k be the k-fold branched cyclic cover of (B^4,D), and take k to be a prime-power. Then $\tilde{H}_*(W_k;\mathbb{Q}) = 0$ (see §5), so, by duality, $H_1(M_k)$ has order ℓ^2, where $G = \ker(H_1(M_k) \longrightarrow H_1(W_k))$ has order ℓ. Note that G has the property, intrinsic to M_k, that the linking form $H_1(M_k) \times H_1(M_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$ vanishes on G. Let V be the closure of the complement of a tubular neighbourhood of D in B^4, and write V_k for the k-fold cyclic cover of V, $1 \leq k \leq \infty$. Then $\partial V_k = N_k$.

Let χ be a character of prime-power order m on $H_1(M_k)$, such that $\chi(G) = 1$. There is then a character $\bar{\chi}$ on $H_1(W_k)$ such that

$$\begin{array}{ccc} H_1(M_k) & \longrightarrow & H_1(W_k) \\ & {\scriptstyle\chi'}\searrow \quad \swarrow{\scriptstyle\bar{\chi}} & \\ & \mathbb{C}^* & \end{array}$$

commutes. Suppose (but only to simplify the exposition) that $\bar{\chi}$ also has order m. Composing with the canonical epimorphism $H_1(V_k) \dashrightarrow H_1(W_k)$, we get a character $\bar{\chi}'$ on $H_1(V_k)$ such that

$$\begin{array}{ccc} H_1(N_k) & \longrightarrow & H_1(V_k) \\ & {\scriptstyle\chi'}\searrow \quad \swarrow{\scriptstyle\bar{\chi}'} & \\ & C_m & \end{array}$$

commutes. We can therefore use V_k to compute $\tau(K,\chi)$. But it can be shown that since V is a homology circle and m is a prime-power, $H_*(\tilde{V}_\infty;\mathbb{Q})$ is finite-dimensional. In particular, $H_2(\tilde{V}_\infty)$ is $\mathbb{Z}[C_\infty]$-torsion. Since $\mathbb{C}(t)$ is flat over $\mathbb{Z}[C_m \times C_\infty]$, it follows that $H_2^t(V_k;\mathbb{C}(t)) = H_2(\tilde{V}_\infty) \otimes_{\mathbb{Z}[C_m \times C_\infty]} \mathbb{C}(t) = 0$, and therefore $w(V_k) = 0$. Again, since V is a homology circle and k is a prime-power, $H_2(V_k;\mathbb{Q}) = 0$ (see §5). Hence $w_0(V_k) = 0$ also, giving $\tau(K,\chi) = 0$.

The vanishing of $\tau(K,\chi)$ for certain characters χ is therefore a necessary condition for K to be slice. To utilize this condition, we first define a

If, in addition, K is slice, and χ satisfies the conditions described earlier which then imply that $\tau(K,\chi) = 0$, we obtain

$$|\sigma(M_k, \chi)| \leq 1 .$$

Since the invariant $\sigma(M_k, \chi)$ can often be calculated, this is a workable condition. For example, if K is a 2-bridge (or rational) knot, and $k = 2$, then M_k is a lens space, and $\sigma(M_k, \chi)$ can be calculated fairly easily using the G-signature theorem. Also, in this case, $\tilde{M}_k$ will always be a rational homology sphere, so K can be slice only if (for suitable χ) $|\sigma(M_k, \chi)| \leq 1$. From this it can be shown that a large number of 2-bridge knots K have $\psi([K]) = 0$ in $W_S(\mathbb{Z})$, but are not slice knots.

14. 3-Manifolds and Knots

In this section and the next we shall discuss some of the functions

$$\{\text{knots}\} \longrightarrow \{\text{3-manifolds}\}$$

which may be defined. Such a function relates knot theory to the general theory of 3-manifolds, and hence by means of it any development in one theory will have consequences for the other. Here, among other things, we shall look at some of the ways in which general results about 3-manifolds have had implications for knot theory. Possible influences in the other direction will be considered in §15.

Probably the most obvious function of the above type is the one which simply associates to a knot its exterior. (This is not known to be injective, but the odds seem good that it is.) Here, as we have already mentioned, Dehn's lemma implied that $\pi K \cong \mathbb{Z}$ only if K is trivial, the sphere theorem implies the asphericity of knots, and Waldhausen's work implies that knots are classified by the triples $(\pi K, \lambda, \mu)$.

We might also mention the fibration theorem of Stallings [142], (see also [111]) which, when applied to knot exteriors, implies that

$M = H_+ \cup_\partial H_-$, say, with $\partial H_\pm$ connected, and $H_\pm \cong \widetilde{B} \times I$, where $\widetilde{B}$ is the corresponding branched cover of (B^2, P). If the projection $M \longrightarrow S^3$ is k-sheeted away from the branch set, and m-sheeted over K, then

$$\chi(\widetilde{B}) = k\,\chi(B^2) - (k-m)\chi(P) = mb - (b-1)k .$$

It follows that $H_\pm$ is a solid handlebody of genus $(b-1)k - mb + 1$, giving a Heegaard splitting of M of that genus. For the k-fold branched cyclic cover, the genus is $(k-1)(b-1)$. In particular, for the 2-fold branched cover, we just get $b-1$. In this way, knots of increasing complexity are mapped to 3-manifold decompositions of increasing complexity.

Now it is known that the 2-fold branched covering function is not injective; many examples of pairs of prime knots with the same 2-fold branched cover are described in [11]. It is injective, however, on the set of 2-bridge knots. There, the 2-fold branched cover has genus 1, and is therefore a lens space, and Schubert has proved [132] that this lens space determines the knot. This injectivity already fails for 3-bridge knots [11]. It has been shown by Birman-Hilden [10], however, as a consequence of a rather special feature of the group of isotopy classes of homeomorphisms of a closed surface of genus 2, that if we regard the 2-fold branched covering function as a function {knots} $\longrightarrow$ {equivalence classes of Heegaard splittings of 3-manifolds}, then it <u>is</u> injective on the set of 3-bridge knots.

Finally, in this context we might mention the result of Waldhausen [157], which says that only the unknot has S^3 as its 2-fold branched cover.

15. Knots and 3- and 4-Manifolds

Continuing in the general framework of §14, let us now consider the possibility of using knowledge about knots to give information about 3-manifolds. In particular, functions {knots} $\longrightarrow$ {3-manifolds} which are surjective, or at least have a sizeable image, will be of interest.

homology spheres, in particular, the dodecahedral space discovered earlier by Poincaré, could be obtained in this way from torus knots. Indeed, the Property P conjecture (see §2) is that if K is non-trivial and $\alpha \neq 0$, then $M(K;1/\alpha)$ is never simply-connected.

It seems likely that the function $M(\_;\beta/\alpha)$ is never injective, although this has only been verified for certain β/α [53], [87]. However, it may not be unreasonable to conjecture that, denoting the unknot by 0, and excluding the trivial case $\alpha = 0$, $M(K;\beta/\alpha) \cong M(0;\beta/\alpha)$ only if $K = 0$. The case $|\beta| = 1$ is just a weakened form of the Property P conjecture, and the case $\beta = 0$ has also received some attention (under the name 'Property R').

Turning to the question of surjectivity, clearly the most one could hope to obtain in this way is the set of all closed, orientable 3-manifolds M with $H_1(M)$ cyclic. This seems highly unlikely. In particular, it is surely not true that all homology spheres can be obtained by Dehn's original method, although this is apparently rather difficult to prove.

The a priori restriction on the homology disappears if one allows, instead of knots, links with arbitrarily many components, and it is indeed the case that one can now obtain all closed, orientable 3-manifolds. Actually a stronger statement is possible. If L is a framed link in S^3, then (ordinary) framed surgery on S^3 along L gives a 3-manifold M(L), say. Wallace [159] and Lickorish [86] have shown that this function {framed links} $\longrightarrow$ {closed, orientable 3-manifolds} is surjective. Wallace's proof is essentially 4-dimensional; it uses the theorem of Rohlin [126] that 3-dimensional oriented cobordism $\Omega_3 = 0$, together with handlebody techniques. (The argument is: given M, there exists W such that $M = \partial W$; W has a handle decomposition with one 0-handle and no 4-handles. Replace the 1- and 3-handles by 2-handles ('handle trading'), giving W'. The attaching maps of the 2-handles in W' now define a framed link L with $M \cong M(L)$.) Lickorish's proof, on the other hand, is 2-dimensional, in the sense that it is based on the fact that the group of isotopy classes of orientation-preserving homeomorphisms of a closed surface is generated by 'twists'. (This was first proved by Dehn [29].)

proofs of this (assuming $\Omega_4 \cong \mathbb{Z}$) have been given by Casson and (independently) Matsumoto (both unpublished), and link-theoretic ideas are involved in these proofs.

Many questions concerning the existence of certain surfaces in 4-manifolds are equivalent, or closely related, to questions about knot and link concordance. Thus Tristram [148] used his p-signatures to show that a class $ax+by$ in $H_2(S^2 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ can be represented by a smoothly embedded 2-sphere only if a and b are coprime. (It is still unknown whether this condition is sufficient, except for the cases $|a| \leq 1$ or $|b| \leq 1$.) As we have seen in §12, signatures of knots (and links) are probably best studied from a 4-dimensional point of view anyway, so this kind of connection is not surprising.

Perhaps more surprising is the result of Casson (unpublished) that simply-connected surgery is possible in dimension 4 if each of a certain explicit set of infinite sequences of links contains a slice link. On the other hand, failure of the latter condition implies the existence of some kind of pathology in dimension 4. For example, if the sequence of (untwisted) doubles of the Whitehead link contains no slice link, then there is a 4-manifold proper homotopy equivalent to $S^2 \times S^2$-point whose end is not diffeomorphic to $S^3 \times R$, and a 4-dimensional counterexample to the McMillan cellularity criterion. (These results are also due to Casson.)

16. Knots and the 3-Sphere

All the abelian algebra discussed so far is valid for knots in homology 3-spheres. Similarly, all known knot concordance invariants are actually homology-cobordism invariants. The group of a knot in S^3, of course, has weight 1 (being generated by the conjugates of any meridian element), but again this is true of a knot in any homotopy 3-sphere. Still, it is clear that the theory of knots in the 3-sphere, having the concreteness and immediacy of the physical world, is of prime importance. Moreover, even properties which hold in more general settings might be more easily observed in the 3-sphere. This has certainly been the case historically. For example, the property $\Delta(1) = 1$ of the Alexander polynomial was first proved by means of knot projections [2]. (In fact the purely combinatorial

17. Other Topics

Here we briefly mention one or two topics which we shall not be able to discuss in detail.

First, there is the whole question of symmetries of knots, For S^1-actions, the answer is known: only the unknot can be the fixed-point set of an S^1-action on S^3, and the only knots which are invariant under (effective) S^1-actions are the torus knots. (This follows from the theory of Seifert fibre spaces [134]; see [65].) For the case of $\mathbb{Z}/p$-actions on S^3 fixing a knot K, we of course have the Smith conjecture that K must be trivial. This is surely one of the major unsolved problems in knot theory. It is known to be true for $p = 2$ [157], and there exist various other partial results, including [16], [42], [45], [50], [56], [109]. Necessary conditions are given in [149] and [108], for a knot K to have a symmetry of order n in the sense that there is a homeomorphism h of S^3 of period n, with fixed-point set a circle disjoint from K, such that $h(K) = K$.

Given an unoriented knot K in oriented S^3, one can ask whether or not there exists an orientation-reversing homeomorphism of S^3 taking K to itself, (or equivalently, an orientation-preserving homeomorphism of S^3 taking K to its mirror-image). If there is, K is amphicheiral. If K is now oriented, one can ask whether there is an orientation-preserving homeomorphism of S^3 taking K onto K but reversing its orientation. If so, K is invertible. If K is amphicheiral, then, for example, all its branched covers will support orientation-reversing homeomorphisms. Because of this, amphicheirality is often relatively easy to detect [135]. Since many knot invariants are independent of the orientation of the knot, however, it is harder to establish non-invertibility. This was first done in [151], by analysing automorphisms of the group. See [71], [161] for further results. Two interesting conjectures relating these concepts to symmetries (see [75]) are: K is amphicheiral if and only if K is invariant under reflection through the origin (van Buskirk); and: K is invertible if and only if there is an orientation-preserving involution of S^3 taking K to itself, reversing its orientation (Montesinos). Apparently these are true for knots with small crossing number.

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