BRAIDS

DALE ROLFSEN — U.B.C.

(joint with R. FENN and JUN ZHU)

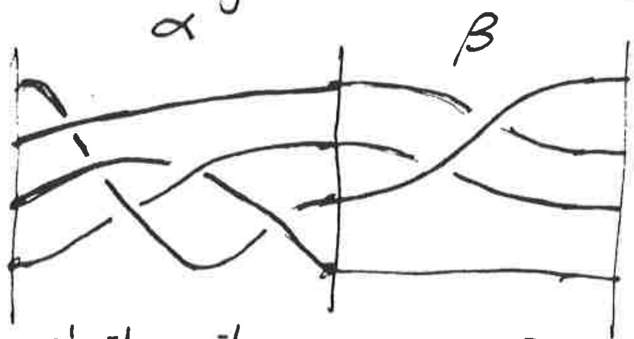
REFERENCES:

Fenn, Rolfsen, Zhu, Centralisers in braid groups and singular braid monoids, L'enseign. Math. 95.

Rolfsen, Normalisers and commensurators in braid subgroups and induced representations, to appear, Invent. Math.

BRAIDS

FOR FIXED $n \geq 1$, the set of braids with n-strings forms a group, B_n

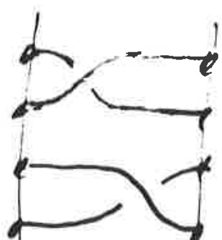


$= \alpha\beta$;

$= \underline{1}$

$\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 \qquad \sigma_2^{-1}\sigma_3^{-3}$

Isotopic braids (with ends fixed) considered equal: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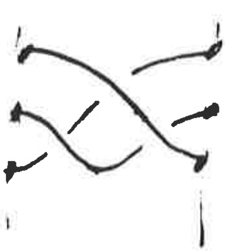
$\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j|>1$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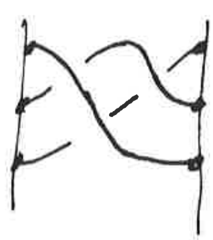


$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

THESE ARE A COMPLETE SET OF RELATIONS (ARTIN)

GENERATORS $\sigma_1, \ldots, \sigma_{n-1}$

EACH BRAID DETERMINES A PERMUTATION OF $\{1, \ldots, n\}$. IF THE PERMUTATION IS TRIVIAL, THE BRAID IS PURE.

The PURE BRAID subgroup P_n is THE KERNEL OF THE MAP $B_n \to \Sigma_n$, i.e.

$$1 \to P_n \to B_n \to \Sigma_n \to 1 \quad \text{EXACT.}$$

OTHER VIEWPOINTS:

BRAID = TIME HISTORY OF n UNDISTINGUISHABLE PARTICLES ROAMING IN DISK D^2 (OR $\mathbb{C}$)

PURE BRAID = TIME HISTORY OF DISTINGUISHABLE PARTICLES.

$$P_n = \pi_1\left(\mathbb{C}^n \setminus \left\{\begin{smallmatrix}\text{big}\\ \text{diagonal}\end{smallmatrix}\right\}\right)$$

$$B_n = \pi_1\left(\widehat{\mathbb{C}^n \setminus \left\{\begin{smallmatrix}\text{big}\\ \text{diag.}\end{smallmatrix}\right\}}\right) \leftarrow \Sigma_n \text{ ORBITS}$$

FADELL-NEUWIRTH: $\mathbb{C}^n \setminus \{\text{big diag.}\}$ HAS

TRIVIAL HIGHER HOMOTOPY

LIKEWISE $\mathbb{C}^n \setminus \{\text{big diag}\}$, SO IT'S $K(B_n, 1)$ (EILENBERG-~~MACLANE~~ SPACE)

B_n HAS FINITE COHOMOL. DIMENSION (in fact C.D. = n-

$\Rightarrow$ B_n HAS NO TORSION.

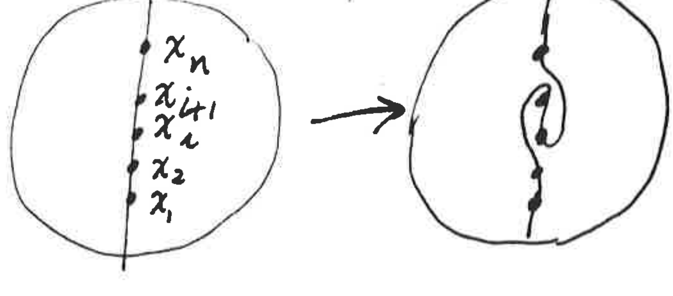
ALSO ~~NOT~~ A 3-MANIFOLD GROUP, $n>4$

MAPPING CLASS VIEW:

$$B_n \cong \pi_0\big(\mathrm{Diff}_\partial(D^2, \underbrace{\{x_1, \dots, x_n\}}_{\text{fixed setwise}})\big)$$

as groups.

σ_i IS REPRESENTED AS:



AT π_1 LEVEL, FREE GROUP $F_n \cong \pi_1(D^2 \setminus \{x_1, .. x$

$B_n \hookrightarrow \mathrm{Aut}(F_n)$

THEREFORE B_n IS <u>RESIDUALLY FINITE</u>

$\forall \beta \overset{\neq 1}{\in} B_n$, $\exists$ FINITE GROUP G AND HOMOMORPHISM

$B_n \xrightarrow{f} G$ WITH $f(\beta) \neq 1$.

(BAUMSLAG: AUT (f.g. res. finite) is <u>RESID. FINITE</u>)

$\Rightarrow$ B_n IS <u>HOPFIAN</u> (NOT $\cong$ A PROPER QUOTIENT)

(SAME FOR P_n)

THURSTON: B_n IS <u>AUTOMATIC</u>

$\Rightarrow$ ALGORITHMIC SOLUTION OF WORD PROBLEM & CONJUCACY PROBLEM

II. BRAID SUBGROUPS

- COMMUTATOR SUBGROUP. If generators commute, then $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $\Longrightarrow \sigma_i = \sigma_{i+1}$. So

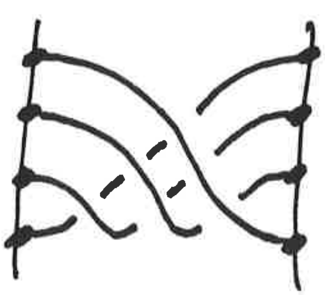
$$\{B_n\}_{ab} \cong \mathbb{Z}$$

The abelianization map is

$$\sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k} \longmapsto \sum_{j=1}^{k} \varepsilon_j$$

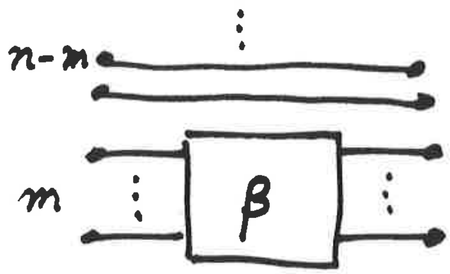
$[B_n, B_n] = \{$ words of degree 0 in $\sigma_i \}$

- CENTRE OF B_n? [Chow, 1948]


$=: \Delta$

For $n \geq 3$, $Z(B_n) \cong \mathbb{Z}$

generator: Δ^2



CONVENTIONAL WAY TO INCLUDE B_m IN B_n, $m < n$.

- $B_m < B_n$ "USUAL INCLUSION"

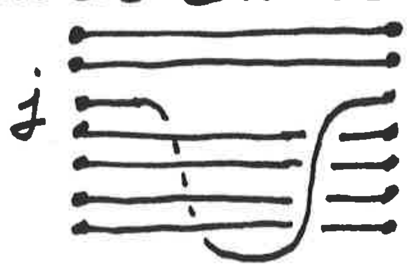
AS SUBGROUP GEN BY $\sigma_1, \ldots, \sigma_{m-1}$
<u>NOT</u> NORMAL.

Q: NORMALISER? (de la Harpe) CENTRALISER? (JONES)

NEW <u>THEOREM</u>: THE <u>CENTRALISER</u> $Z_{B_n}(B_m)$

IS THE SUBGROUP OF B_n GENERATED BY:

- $\Delta^2 = \Delta_m^2$ (generates <u>centre</u> of B_m)
- $\sigma_j \quad j > m$
- $\rho_j \quad j > m$



ρ_j

<u>THEOREM</u>: THE <u>NORMALISER</u> $N_{B_n}(B_m)$ IS THE GROUP GENERATED BY B_m AND THE CENTRALISER:

$$N_{B_n}(B_m) = \langle B_m, Z_{B_n}(B_m) \rangle.$$

MOREOVER, B_m IS A DIRECT SUMMAND.

- COMMENSURATOR?

<u>DEF</u>: SUBGROUPS $A, B < G$ ARE <u>COMMENSURABLE</u> IF $[A : A \cap B] < \infty$ & $[B : A \cap B] < \infty$.

IF $H < G$, $\mathrm{Com}_G(H) :=$

$$\{ g \in G \mid g^{-1} H g \text{ AND } H \text{ COMMENSURABLE} \}$$

EXAMPLE: $G = B_n \quad H = \langle \Delta_n \rangle$

THEN $\quad Com_{B_n}(H) = B_n \quad$ ($\langle \Delta^2 \rangle$ FIXED UNDER CONJ.)

BUT $\quad N_{B_n}(H) = \{$ words invariant under $\sigma_i \longleftrightarrow \sigma_{n-i} \}$

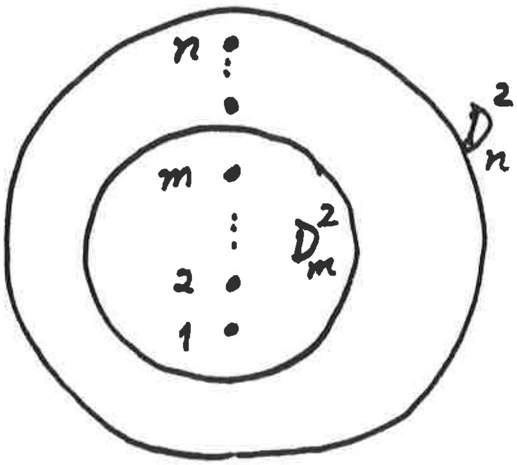
NEW

THM: $\quad Com_{B_n}(B_m) = N_{B_n}(B_m)$

KEY REASON: $\beta \leftrightarrow \sigma_i \quad$ IFF

$\beta \leftrightarrow \sigma_i^N \quad$ some $N \neq 0$.

AS <u>MAPPING</u> <u>CLASS</u> <u>GROUPS</u> $B_m \hookrightarrow B_n$ IS INDUCED BY THE INCLUSION $D_m^2 \subset D_n^2$



UNDER THE ISOMORPHISM

$$B_n \cong \pi_0 \underline{\mathrm{Diff}_\partial}(D^2_n, \{1, \dots, n\})$$

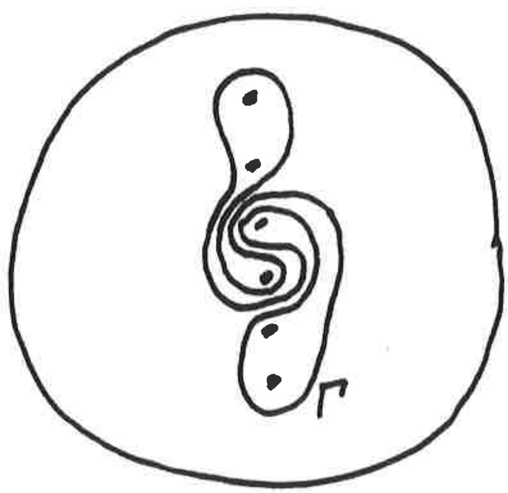
$\leftarrow \partial \Gamma_m$

$$B_m \cong \mathrm{Stab}(D^2_n \setminus D^2_m)$$

$$Z_{B_n}(B_m) \cong \mathrm{Stab}(D^2_m)$$

$$N_{B_n}(B_m) = \mathrm{Stab}(\Gamma_m) \qquad \Gamma_m = \partial D^2_m$$

'GEOMETRIC' BRAID SUBGROUPS



$B_\Gamma \cong$ Diffeomorphisms fixed outside Γ.

CURVES IN $D^2_n \setminus \{1,\dots,n\}$, UP TO ISOTOPY,

PARAMETRIZE THE GEOMETRIC SUBGROUPS

$B_\Gamma = \mathrm{Stab}(\text{outside of } \Gamma)$

THEN $Z_{B_n}(B_\Gamma) = \mathrm{Stab}(\text{inside of } \Gamma)$

AND $N_{B_n}(B_\Gamma) = \mathrm{Stab}(\Gamma)$

Each B_Γ is conjugate to some B_m, where

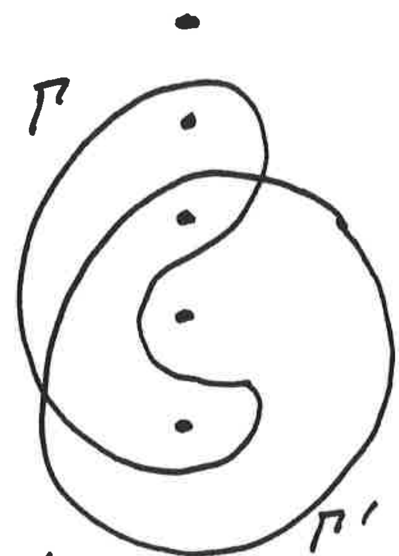
m = number of points enclosed by Γ.

$\Gamma = g\Gamma_m \Rightarrow \mathrm{stab}(\Gamma) = g\,\mathrm{stab}(\Gamma_m)\,g^{-1}$

"GENERIC" CASE: $1 < m < n-1$

<u>THEOREM</u>: For generic Γ, Γ' the stabilisers $\mathrm{Stab}(\Gamma)$ and $\mathrm{Stab}(\Gamma')$ are <u>COMMENSURABLE</u> iff $\Gamma = \Gamma'$ (up to isotopy in $D^2_n \setminus \{1, \dots, n\}$).

<u>PROOF.</u>



$i(\Gamma, \Gamma') =$ min intersection up to isotopy in $D^2_n \setminus \{1, \dots, n\}$

<u>IF</u> $\Gamma \neq \Gamma'$

CHOOSE Γ'' with $i(\Gamma'', \Gamma') \neq 0$, $i(\Gamma'', \Gamma) = 0$

$[g] \in \pi_0 \mathrm{Diff}_\partial (D^2_n, \{1, \dots, n\})$ where

$g =$ DEHN TWIST ALONG Γ''

$\{g^k\}$ IS AN INFINITE SUBGROUP OF $\mathrm{Stab}(\Gamma)$ WHICH INTERSECTS $\mathrm{Stab}(\Gamma')$ ONLY IN $\{1\}$.

$\Rightarrow$ $\mathrm{Stab}(\Gamma') \cap \mathrm{Stab}(\Gamma)$ <u>INFINITE</u> INDEX IN $\mathrm{Stab}(\Gamma)$ $\Rightarrow$ <u>IN</u>COMMENS.

<u>COR</u>: For $B_m < B_n$ OR $B_\Gamma < B_n$ ANY GEOMETRIC SUBGROUP, THE <u>COMMENSURATOR</u> IS <u>SELF-COMMENSURATING</u>.

$$\mathrm{Com}_{B_n}(\mathrm{Com}_{B_n}(B_\Gamma)) = \mathrm{Com}_{B_n}(B_\Gamma)$$

III. OPEN QUESTIONS:

- Is B_n LINEAR? (FINITE-DIMENSIONAL FAITHFUL REP.?)
- Does its group ring $\mathbb{Z} B_n$ have zero divisors? Nontrivial units? (conjecture: NO, even under the weaker hypothesis: torsion-free fin. gen.)

ORDERED GROUPS.

If G can be totally ordered by $<$, respecting multiplication:

$$x < y \Rightarrow xz < yz \text{ and } zx < zy,$$

it is called '<u>orderable</u>.'

THEOREM: P_n IS ORDERABLE.

COR: $\mathbb{Z}P_n$ HAS NO ZERO DIVISORS OR TRIVIAL UNITS.

PROOF OF THEOREM: Follows from Falk-Randall result that lower central series has <u>free</u> abelian quotients: $\dfrac{(P_n)_k}{(P_n)_{k+1}}$ & nilpotent, torsion-free, fin. gen $\Rightarrow$ orderable.

& $\bigcap_1^\infty (P_n)_k = 1$

PROP: B_n IS NOT ORDERABLE, $n > 2$.

REASON: GENERALISED TORSION

Def: g is a gen. torsion element if $g \neq 1$

and $\exists\, x_1, \ldots, x_k \in G$ with

$$g\,(x_1^{-1} g x_1) \cdots (x_k^{-1} g x_k) = 1.$$

PROP: Orderable $\Rightarrow$ NO GENERALISED TORSION

But B_3 HAS generalised torsion!

Let $g = \sigma_1 \sigma_2^{-1}$, $\Delta = \Delta_3 = \sigma_1 \sigma_2 \sigma_1$

THEN $\Delta^{-1} g \Delta = \sigma_2 \sigma_1^{-1} = g^{-1} \Rightarrow \underline{g(\Delta^{-1} g \Delta) = 1}$

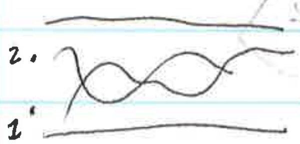
$\Rightarrow g$ is a generalised torsion elt.

~~Assume~~ $\cancel{X = K(B_n,1)}$

Braids: Dale Rolfsen

<u>References</u>: Centralizers in braid groups ...... L'enseignement Math 95

Rolfsen To appear Inventiones

n-strings  σ_2  $B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \ |i-j|>1,$
$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$

<u>Other views</u>

(1) Braid = Time History of n undistinguished pts in D^2 ending at same set

(2) $P_n = \pi_1(\mathbb{C}^n - \text{big diagonal})$ $B_n = \pi_1(\overbrace{\mathbb{C}^n - D}^{\Sigma_n\text{-orbits}})$

Fadell–Neuwirth: $\mathbb{C}^n - D$ is aspherical and also

$\Rightarrow \left\{ \begin{array}{l} B_n \\ P_n \end{array} \right.$ has finite coh. dim and is torsion-free

(3) $B_n = \pi_0(\mathrm{Diff}_\partial(D^2, \{x_1, \dots, x_n\}))$ as groups
fixed setwise

(4) $B_n \hookrightarrow \mathrm{Aut}(F_n)$ $F_n = \pi_1(D^2 - \{x_1, \dots, x_n\})$

$A \hookrightarrow G \twoheadrightarrow F$

Question:? Does $\exists$ faithful rep. for every n of B_n in Linear group

(Baumslag)

FACT: Free groups are residually finite, $\mathrm{Aut}(\text{f.g. r.f.})$ is residual finite

preserved under subgroups $\Rightarrow B_n$. resid. finite $\Rightarrow$ Hopfian

$\boxed{\text{Thurston}}$: B_n is automatic

Rolfsen (2)

• Def: 2 subgroups $A, B \leq G$ are commensurable if $[A; A\cap B] < \infty$
$[B, A\cap B] < \infty$

if $H < G$

Def: $Com_G(H) = \{g \in G \,/\, gHg^{-1}$ and H are commensurable$\}$

eg. $G = B_n$ $H = \langle \Delta_n \rangle$ then $Com_{B_n}\langle H \rangle = B_n$ Δ^2 fixed under conj!!

exercise → but $N_{B_n}(H) =$ words invt under $\sigma_i \to \sigma_{n-i}$

Thm $Com_{B_n}(B_m) = N_{B_n}(B_m)$ Key: σ comm $\sigma_i \Leftrightarrow \sigma$ comm σ_i^N some $N \neq 0$

∃ "Abstract" Commensurator of a group

$B_m = Stab(D_n^2 - D_m^2)$ pointwise

$Z_{B_n}(B_m) = Stab\, D^2_m$ pointwise

$\Gamma_m = \partial D_m^2$

$N = stab(\Gamma_m)$ pointwise !!!

(3) Rolfsen

Thm: FALK-Randall: ~~[illegible]~~

P_n satisfies these $\Rightarrow$ orderable

?? proper non-abelian quotient of Braid B_n which is torsion-free