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Link Concordance and Groups

by

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Abstract

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This work concerns the study of link concordance using groups, both extracting concordance data from group theoretic invariants and determining the properties of group structures on links modulo concordance. Milnor's invariants are one of the more fundamental link concordance invariants; they are thought of as higher order linking numbers and can be computed using both Massey products (due to Turaev and Porter) and higher order intersections (due to Cochran). In this thesis, we generalize Milnor's invariants to knots inside a closed, oriented 3-manifold M. We call this the Dwyer number of a knot and show methods to compute it for null-homologous knots inside a family of 3-manifolds with free fundamental group. We further show Dwyer number provides the weight of the first non-vanishing Massey product in the knot complement in the ambient manifold. Additionally, we prove the Dwyer number detects knots K in M bounding smoothly embedded disks in specific 4-manifolds with boundary M which are not concordant to the unknot. This result further motivates our definition of a new link concordance group using the knotification construction of Ozsvàth and Szabò. Finally, we give a proof that the string link concordance group modulo its pure braid subgroup is non-abelian.

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For my sister Malak.

here is the deepest secret nobody knows
(here is the root of the root and the bud of the bud
and the sky of the sky of a tree called life;which grows
higher than soul can hope or mind can hide)
and this is the wonder that's keeping the stars apart

i carry your heart(i carry it in my heart)

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e.e. cummings

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1.1 Introduction for non-mathematicians

Imagine standing in a flat, empty, rectangular field. You can walk from any place to any place, and the fastest way to do it would be a straight line. If you had to, and you had a smartphone to remind yourself how to do it, you could even figure out exactly how fast you were going while walking from one place on the field to another.

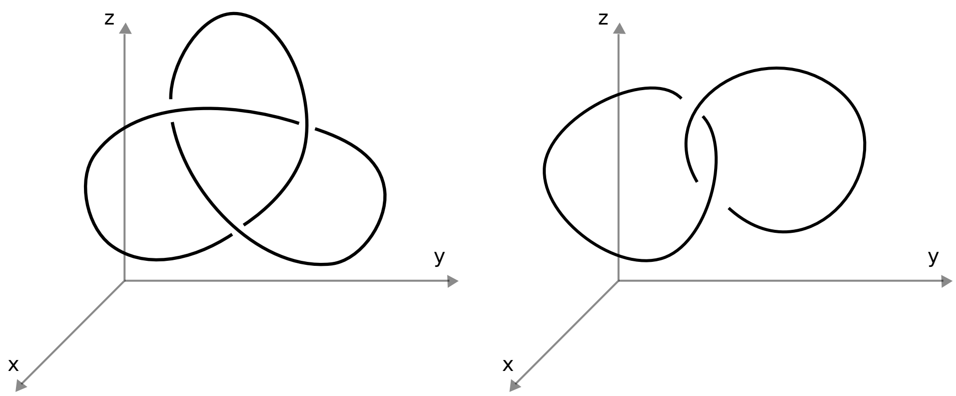
Now, imagine instead that you are standing in a maze. Without a bird's eye view, it would be very difficult to tell whether it was possible to walk from any point in the maze to any other, and what the quickest route would be. If you could understand more about the shape of the maze, even just a few hints, you would be better able to tell if you could walk from one point to another and whether you could do so in a reasonable amount of time.

The study of the shape of spaces is called topology. When we become concerned with measuring distances and angles, that is an example of geometry. Just like in the examples of an empty field and a maze, before you can understand how to move in a space or more generally use a space (geometry) you should understand its general features (topology). In fact, there is even evidence that our brains store information about our surroundings topologically [**?**].

Determining when two spaces have "the same shape" is difficult; this difficulty depends on the dimension of the spaces involved. Many properties can be proven for spaces of dimensions 5 and higher, while dimensions 1, 2, and 3 are can be understood well through other methods. 4-dimensional spaces can be thought of as the "bridge" between low-dimensional behavior and high-dimensional behavior and much about it is still unknown. Furthermore, we live in a (at least) 4-dimensional world: there are three spatial dimensions and one time dimension. Thus, understanding what

4-dimensional spaces are possible and how to identify them could prove helpful to better understanding the world around us.

It is perhaps a surprising fact that knot theory provides a useful tool for studying 3- and 4-dimensional spaces. A knot is a smooth, closed circle sitting in 3-dimensional Euclidean space, and an n-component link is a collection of n smooth, closed circles sitting in 3-dimensional space which do not intersect each other as in Figure 1.



(a) A knot. (b) A 2-component link.

Figure 1: An example of a knot and link in 3-dimensional space.

A theorem of Lickorish and Wallace shows every 3-dimensional shape which is "nice" in some technical sense is obtained by a certain operation called surgery on a link; in this sense, studying knots and links can be viewed as studying something fundamental about 3-dimensional space. Moreover, we can algorithmically construct every 4-dimensional shape we could have a hope of doing geometry on using an operation called attaching 4-dimensional "handles" along a link [**?**]. In this way, we can organize relationships between 3- and 4-dimensional space using an algebraic structure called the knot concordance group and study it using tools from knot theory.

In this thesis, I develop a new tool called the Dwyer number to detect whether knots in a specific, more complicated family of 3-dimensional shapes are "the same" when considered up to a 4-dimensional relation called concordance. Furthermore, I analyze the properties of Dwyer number and use it to show there are infinitely many knots which are distinct up to this concordance relationship, but look as simple as possible when viewed in different, but related 4-dimensional shape. Additionally, I use more classical tools called Milnor's invariants (which the Dwyer number generalizes) in order to show that a different algebraic structure created from links is more complicated than previously thought.

1.2 Introduction for mathematicians

This thesis concerns low-dimensional topology; more specifically, this body of work is concerned with how algebraic structures associated to a topological space can classify the smooth topology of the space. Whether algebra can completely classify a space depends on dimension; while every n-manifold homotopy equivalent to the n-sphere S^n is homeomorphic to S^n due to work of [?, ?, ?, ?, ?], this homeomorphism cannot generally be upgraded to a diffeomorphism [?, ?]. In dimension 4, this problem is known as the smooth 4-dimensional Poincaré conjecture and remains one of the fundamental questions of low-dimensional topology. Many strange phenomena appear in the 4-dimensional world only; for example, $\mathbb{R}^n$ has exactly one smooth structure for $n \neq 4$ while $\mathbb{R}^4$ has infinitely many distinct smooth structures [?].

For this reason, this thesis is primarily about relationships between 3- and 4-dimensional topology. In particular, we use knot theory as a lens with which to approach this topic. Recall that a knot is a smooth (oriented) embedding of S^1 into S^3 and an n-component link is a smooth (oriented) embedding of n-disjoint copies of S^1 into S^3. A link can be associated to a 3-manifold by removing its normal bundle and gluing it back in with a specified framing (called Dehn surgery) and to a smooth 4-manifold by attaching handles; every 3-manifold and smooth 4-manifold arises in this way as outlined in [?]. Therefore knots and links provide a useful avenue to investigate such manifolds.

In order to use knots to understand smooth 4-manifold topology, we examine their equivalence classes up to a 4-dimensional relation called concordance. Two knots K_0 and K_1 are smoothly concordant if there is a smoothly embedded disjoint annulus in $S^3 \times I$ with boundary $K_0 \times \{0\}$ and $K_1 \times \{1\}$. A knot $K \subset S^3$ is slice if it bounds a smooth, properly embedded disk in B^4. This concordance perspective has useful implications for the study of 4-manifolds; for example, [**?**] proves that showing the Whitehead double of the Borromean rings is not freely topologically slice would disprove the surgery conjecture for 4-manifolds with free fundamental group. Knots modulo concordance with the operation connected sum forms the knot concordance group $\mathcal{C}$ introduced by Fox and Milnor in [**?**].

The main goal of this work is to understand the interaction between 3- and 4-manifold topology using groups of knots and links modulo concordance. We examine the relationship between classical invariants derived from the quotients of an associated fundamental group by its lower central series and a modern construction from Heegaard Floer homology to better classify 3- and 4-manifold topology. Later, we extend the knot concordance group to a group of knots in connected sums of $S^2 \times S^1$. Finally, we show that a previous notion of concordance group of links called the string link concordance group is non-abelian even when its quotient is taken by the normal closure of the pure braid group, indicating that this group is more complex than previously thought.

1.3 Summary of results

Milnor's $\bar{\mu}$-invariants provide a way to classify the relationship between a link L in a homology 3-sphere M and the lower central series quotients of the fundamental group of the link complement; they can be contextualized as higher order linking numbers and have previously been generalized for a small class of homotopically nontrivial knots inside prime manifolds and Seifert fiber spaces [?, ?]. $\bar{\mu}$-invariants are notoriously difficult to compute for even simple links; in practice, it is usually only possible to compute the first non-vanishing Milnor invariants. This subset of $\bar{\mu}$-invariants can be viewed as determining how deep the based homotopy classes of longitudes of L are in the lower central series of the link group $\pi_1(M \setminus \nu(L), *)$ where $\nu(L)$ is a regular neighborhood of L.

In order to capture similar higher order linking data contained in lower central series quotients for knots in arbitrary 3-manifolds, we defined a new concordance invariant which we call the Dwyer number and denote $D(K, \gamma)$ for a knot $K \subset M$ with M a closed, oriented 3-manifold and γ a specific simple closed curve in the same free homotopy class as K. $D(K)$ detects which elements of M can be represented by maps of special 2-complexes called half-gropes described in Section **??**. $D(K)$ can be viewed as a generalization of the first non-vanishing Milnor's $\bar{\mu}$-invariant of a link. In particular, for $K \subset \#^i S^2 \times S^1$ with associated group $G = \pi_1(\#^i S^2 \times S^1 \setminus \nu(K), *)$, $D(K)$ detects how deep a based homotopy class of a longitude of K is in the lower central series G_q.

Definition 1.1. Let M be a oriented, closed 3-manifold and γ be a fixed smooth, embedded curve inside M. Let $[\gamma]$ be its corresponding free homotopy class. Let $K \subset M$ be a smooth knot with free homotopy class $[K] = [\gamma]$. Then the Dwyer number of K relative to γ is

$$D(K, \gamma) = \max \left\{ q \;\middle|\; \frac{\mathrm{H}_2(M \setminus \nu(K))}{\Phi_q(M \setminus \nu(K))} \cong \frac{\mathrm{H}_2(M \setminus \nu(\gamma))}{\Phi_q(M \setminus \nu(\gamma))} \right\}.$$

where $\Phi_q(X)$ denotes the subgroup of $H_2(X)$ consisting of element which can be represented by maps of special 2-complexes called class $q+1$ half gropes. A class $q+1$ half-grope is constructed recursively out of layers of surfaces: the lowest stage (called the second stage for indexing reasons) is a an oriented surface. Exactly half of a symplectic basis for this surface themselves bound oriented surfaces, this process continues until we have q layers of surfaces.

This invariant is particularly simple for the case of $K \subset \#^l S^2 \times S^1$, which as we will see in Section **??**.

Corollary 1.2. *Let $K \subset \#^l S^2 \times S^1$ be a null-homologous knot and γ be an unknot in $\#^l S^2 \times S^1$. In this case, denote $D(K, \gamma)$ by $D(K)$ and see that*

$$D(K) = max \; \{ \; q \;\; | \;\; \frac{H_q(\#^l S^2 \times S^1 \setminus K)}{\Phi_q(\#^l S^2 \times S^1 \setminus K} = 0 \; \}$$

Recall that for a knot K in S^3, K being slice in B^4 is equivalent to K being concordant to the unknot in $S^3 \times I$. In this thesis, we showed there is in an infinite family of knots in connected sums of $S^2 \times S^1$ which are slice in boundary connected sums of $S^2 \times D^2$ but are not concordant to the unknot (or each other) using our invariant $D(K)$. We further showed $D(K)$ is a new invariant carrying the properties enjoyed by the $\bar{\mu}$-invariants of links $L \subset S^3$. Notably, we showed $D(K)$ is an invariant of concordance of knots in $(\#^i S^2 \times S^1) \times I$ and exhibits the correspondence between Milnor's invariants and Massey products proved independently by Turaev in [**?**] and Porter in [**?**].

Theorem 1.3. *If the Dwyer number of a null-homologous knot $K \subset \#^i S^2 \times S^1$ is q and $G = \pi_1(\#^n S^2 \times S^1 \setminus K, *)$, then*

1. *the longitude of K lies in G_q,*
2. *all non-vanishing Massey products of elements of $H^1(\#^i S^2 \times S^1 \setminus K); \mathbb{Z})$ are weight q or higher.*

Additionally, we used the Dwyer number to detect an entire family of knots in $\#^i S^2 \times S^1$ which bound disks in a particular 4-manifold, but are not concordant to the unknot as detailed in Section **??**. This has applications to the definition of a new link concordance group as described in Section **??**.

Theorem 1.4. *For each $i \in \mathbb{Z}_+$ there is a knot $J_i \subset \#^i S^2 \times S^1$ which bounds a smooth, properly embedded disk in $\natural^i S^2 \times D^2$ in but is not concordant to the trivial knot (even stably as in* **??***) .*

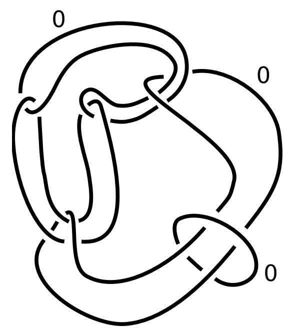


Figure 2: $J_3 \subset \#^3 S^2 \times S^1$

This theorem further motivated the definition of a new notion of link concordance group. Connected sum of knots does not extend naturally to links and defining a link concordance group is somewhat more complicated. In [**?**], Hosokawa defined the group $\mathcal{H}$ of links modulo a specific type of cobordism using the operation of disjoint union instead; however, $\mathcal{H}$ is isomorphic to $\mathcal{C} \oplus \mathbb{Z}$ and therefore does not contain much more structure than the classical knot concordance group. Le Dimet in [**?**] as well as Donald and Owens in [**?**] defined groups by constructing multiple representatives for each link. Le Dimet's string link concordance group $\mathcal{C}(m)$ is not abelian as it contains the pure braid group as a subgroup, while the Donald-Owens group $\mathcal{L}$ is abelian[**?**, **?**]. This indicates that some of their structure is determined by choice of group representative and not by inherent properties of links. In a collaboration with Matthew Hedden, we defined a new group using the knotification construction in [**?**] taking an n-component link $L \subset S^3$ to a unique null-homologous knot $\kappa(L) \subset \#^{n-1} S^2 \times S^1$ called the knotification of L. The group operation is connected sum and we detail the group's construction in Section **??**.

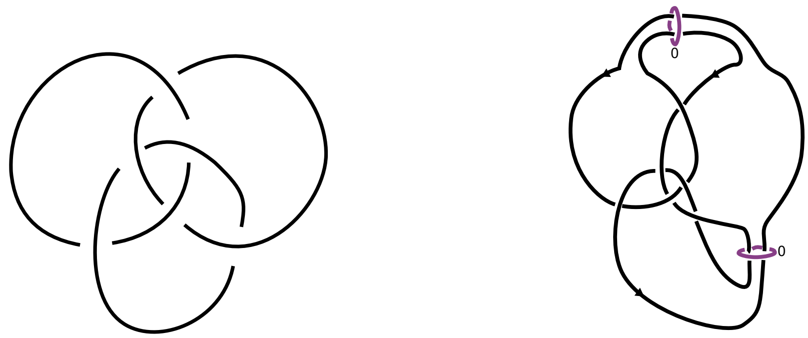


Figure 3: A link $L \subset S^3$ and a surgery diagram for its knotification $\kappa(L) \subset \#^3 S^2 \times S^1$.

Two knotified links are stably concordant they have concordant stabilizations in $(\#^i S^2 \times S^1) \times I$. A knotified link $\kappa(L) \subset \natural^i S^2 \times D^2$ is slice if it bounds a smooth, properly embedded disk in $\natural^i S^2 \times D^2$ as detailed in Section **??**, and the zero elements in the group $\mathcal{C}^{\kappa(L)}$ are exactly slice knots. Though a knot $K \subset S^3$ is slice (trivial in $\mathcal{C}$) if and only if it is concordant to the unknot, we previously showed this is not true for knots in general 3-manifolds using Theorem 1.4. This theorem relies on the Dwyer number; we further showed the Dwyer number of a knotified link is bounded below by a function of the first nonvanishing Milnor invariant of the original link $L \subset S^3$

Theorem 1.5. *The Milnor invariants of a link $L \subset S^3$ give bounds on the Dwyer number $D(\kappa(L))$.*

As a result of this analysis, in order to construct the inverse of a knotified link, one must say two knotified links $\kappa(L_1)$ and $\kappa(L_2)$ are equivalent if the connected sum of their stabilizations $\kappa(L_1)$ and $\kappa(L_2)$ (with the reverse orientation) is slice. We call this stable slice equivalence.

Theorem 1.6. *The set of nullhomologous knots inside connected sums of $S^2 \times S^1$ modulo stable slice equivalence forms an abelian group $\mathcal{C}^{S^2 \times S^1}$ containing the set of* *knotifed links modulo stable slice equivalence $\mathcal{C}^{\kappa(L)}$ as a subgroup. Moreover, $\mathcal{C}^{\kappa(L)}$ contains the knot concordance group $\mathcal{C}$ as a subgroup and concordant links in S^3 become slice equivalent knotified links in $\mathcal{C}^{\kappa(L)}$.*

The knotification $\kappa(L) \subset \#^{n-1}S^2 \times S^1$ of an n-component link $L \subset S^3$ arises naturally in the seminal papers on Heegaard Floer homology [?, ?]. This theory is a rich source of useful but difficult to compute concordance invariants such as Ozsvàth and Szabò's τ-invariant [?], Ozsvàth, Stipsicz, and Szabò's Υ invariant [?], and Jennifer Hom's ϵ invariant [?] which have led to deep results about the knot concordance group. However, many of the link concordance invariants arising in this context factor through knotified links and therefore understanding these objects topologically may lead to important results in Heegaard Floer homology.

Finaly, in additional solo work we returned to previous notions of link concordance group and examined what structure in Le Dimet's string link concordance group $\mathcal{C}(m)$ is inherited from its subgroup of pure braids $\mathcal{P}(m)$ [?, ?]. This quotient is difficult to study as the pure braid group is only normal in the case $m = 2$. Through carefully constructing examples and exploiting a specific type of Milnor invariant called the Sato-Levine invariant, we have shown the following.

Theorem 1.7. *The quotient of the group $\mathcal{C}(m)$ of string links on m strands by the normal closure of the pure braid group $\mathcal{P}(m)$ is non-abelian.*

1.4 Outline of thesis

Chapter 2 contains the necessary background information used in the results in this thesis. In Sections **??** and **??**, we review the necessary background information on knot and link concordance and introduce the knot concordance group. In Section **??** there is further discussion on link concordance groups and a brief survey of the area. Section **??** is survey of Milnor's invariants starting from the original combinatorial group theory definition and ending with the intersection theory perspective from [?] which we use in our results. Finally, Section **??** introduces a few of the main ideas from Heegaard Floer homology.

Chapter 3 contains the bulk of the results in this work. In Section **??** , we introduce a link concordance monoid using a construction from Heegaard Floer and introduce an invariant with which to study this monoid. Then in Section **??**, we introduce a concordance invariant of knots in closed 3-manifolds which generalizes Milnor's invariants using subgroups of second homology generated by half-gropes of a certain height. In Section **??** we prove this invariant can be directly computed for null-homologous knots in connected sums of $S^2 \times S^1$ and exhibits important properties of Milnor's invariants. We further use it to show there is an infinite family of knots in these 3-manifolds which are slice in a certain bounding 4-manifold but are all distinct in concordance. Lastly, in Section **??** we use this result to construct a group from the monoid introduced in Section **??**.

In Chapter 4, we discuss the relationship between string links and the indeterminacy of Milnor's invariants in Section **??**. Then in Section **??**, we prove the string link concordance group modulo the normal closure of the pure braid group is non-abelian using the Sato-Levine invariant (a specific type of Milnor invariant).