A new concordance invariant of knots in sums of $S^2 \times S^1$

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Preliminaries

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Proposition

A knot (or link) $K \subset S^3$ is slice if and only if it is concordant to the unknot (or unlink).

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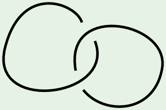

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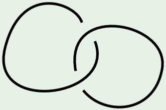

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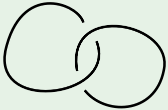

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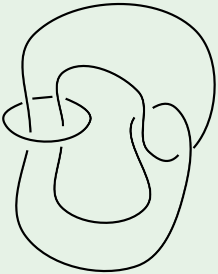

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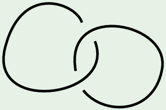

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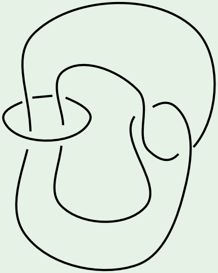

(b) $lk(K, J) = 0$ so ...?

Linking number in the context of groups

Fact:

If L is an n-component oriented link with L_i the 0-framed longitude of the i^{th} component of L and $G = \pi_1(S^3 \setminus \nu(L), *)$, then

$$[L_i] = \Sigma_{i=1}^{n} \mathrm{lk}(L_i, L_j) \cdot x_i \in \mathrm{H}_1(S^3 \setminus \nu(L)) = G/[G, G]$$

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Question:

What if you look at the image of this longitude in a different quotient of G?

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Motivating Idea:

Look at the image of a longitude L_i inside the quotient G/G_q!

Concordance data from the lower central series

Notice:

If $L \subset S^3$ is an n-component link, then $\mathrm{H}_1(S^3 \setminus \nu(L)) = G/[G,G] = \mathbb{Z}^n$ and the n-component unlink has $\pi_1(S^3 \setminus \nu(U), *) \cong F(n)$.

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Rough definition (Milnor '54)

The Milnor invariants of an n-component link $L \subset S^3$ with link group G are a set of integers

$$\overline{\mu}_L(I) \in \mathbb{Z}$$

with $I = (i_1...i_k)$ and $i_j \in \{1, ..., n\}$ detecting when G/G_q stops being isomorphic to F/F_q where F is the rank n free group.

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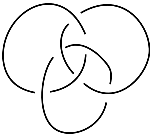
- $\bar{\mu}_L(ijk) =$ triple linking number



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- (Milnor '54) $\bar{\mu}_L(I)$ is a link homotopy invariant for each I with non-repeating indices.

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- (Cochran '90) The first non-zero $\bar{\mu}_L(I)$ (and thus, the first q for which G/G_q is not isomorphic to F/F_q) can be computed using intersection theory.

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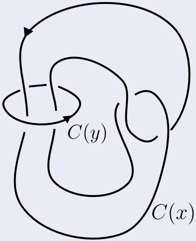

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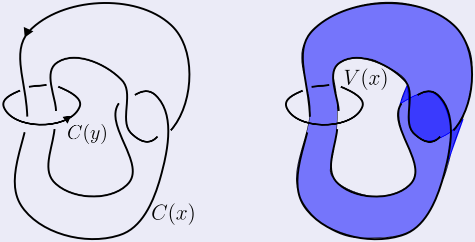

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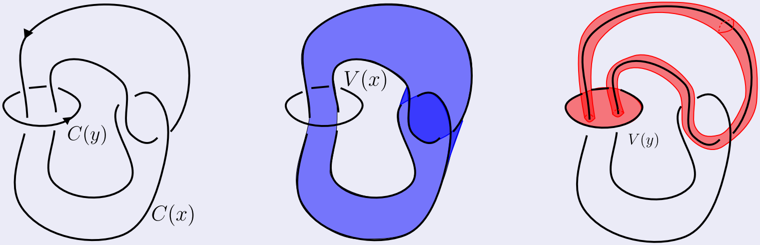

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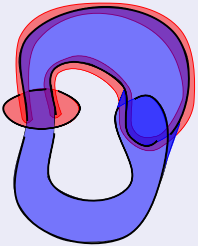

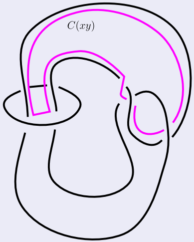

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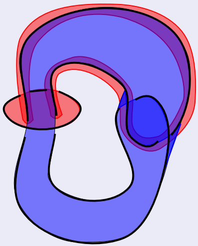

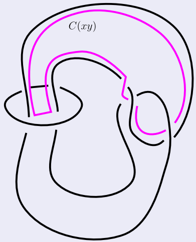

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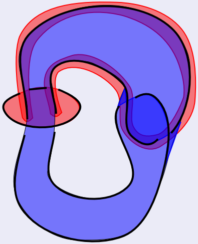
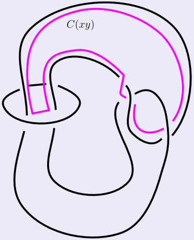

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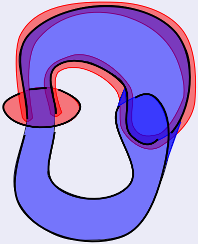

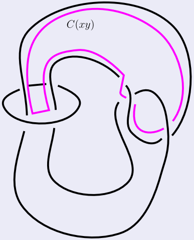

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Is there a version of this linking data for knots or links in other 3-manifolds?

Question:

For a knot or link $L \subset M$ where M is an oriented 3-manifold, can we similarly extract concordance data from quotients of $G = \pi_1(M \setminus \nu(K), *)$ by G_q?

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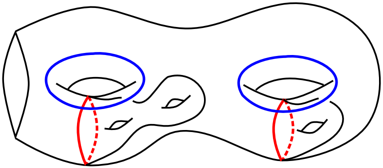
Idea:

Exploit surfaces to define analogue of first non-vanishing $\bar{\mu}_L(I)$!

Realizing iterated commutators geometrically



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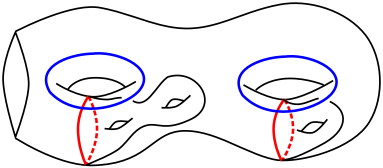

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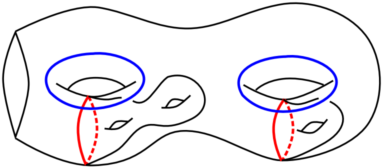

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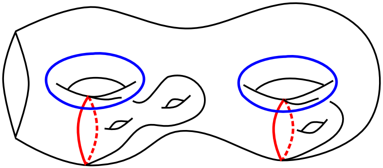

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3. For each i, exactly half of the generators in a symplectic basis for $H_1(\Sigma_3^i)$ ...

The Dwyer number of a knot $K \subset \overset{l}{\#} S^2 \times S^1$

Definition (Dwyer '75, reformulation by Cochran-Harvey '07)

For a space X, $\Phi_n(X) \subset \mathrm{H}_2(X)$ is the subgroup generated by homology classes which can be represented by maps of surfaces which are the first layer of an $n+1$ half-grope.

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Definition (K.)

Let K be a null-homologous knot in $\#^l S^2 \times S^1$. The Dwyer number of K is

$$D(K) = \max \left\{ q \mid \frac{\mathrm{H}_2(\#^l S^2 \times S^1 \setminus K)}{\Phi_q(\#^l S^2 \times S^1 \setminus K)} = 0 \right\}$$

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If K is a null-homologous knot in $\#^l S^2 \times S^1$ with $G = \pi_1(\#^l S^2 \times S^1 \setminus K, *)$, then $D(K) = q$ if and only if G/G_k is isomorphic to F/F_k for $k < q$ and G/G_q is not isomorphic to F/F_q.

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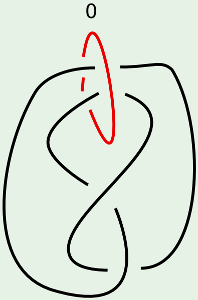
$D(K)$ is an invariant of concordance in $(\#^l S^2 \times S^1) \times I$.

Properties of $D(K)$.

Example



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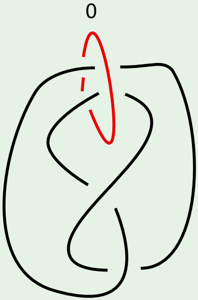

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- If every homology class in $\mathrm{H}_2(\#^l S^2 \times S^1 \setminus K)$ can be represented by a half-grope of arbitrary class, we say $D(K) = \infty$

Properties of $D(K)$.

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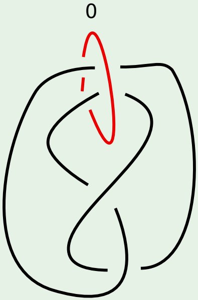

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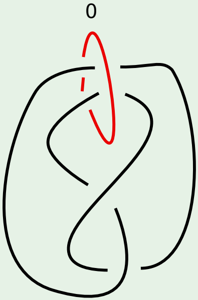

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- $3 \leq D(K) \leq \infty$.

$D(K)$ behaves like first non-vanishing $\overline{\mu}_L(I)$.

Theorem (K.)

If K is a null-homologous knot in $\#^l S^2 \times S^1$ and $D(K) = q$, then the first non-vanishing Massey product in $H^1(\#^l S^2 \times S^1 \setminus K, *)$ is weight q.

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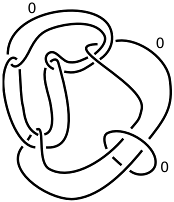
Theorem (K.)

There is an infinite family $\{M_l\}$ of null-homologous knots in $\#^l S^2 \times S^1$ which bound null-homologous disks in $\natural^l S^2 \times D^2$ and distinct in (stable) concordance.

What does this mean?



$K_3 \subset \#^3 S^1 \times S^2$ with $D(K) = 4$

For knots in $K \subset \#^l S^2 \times S^1$,

$$\text{concordance} \implies \text{slice in } \natural^l S^2 \times D^2$$

$$\text{slice in } \natural^l S^2 \times D^2 \not\Longrightarrow \text{concordance.}$$

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For every oriented n-component link $L \subset S^3$ we can construct a knot $\kappa(L) \subset \#^{n-1} S^1 \times S^2$ which is unique up to diffeomorphism of $\#^{n-1} S^1 \times S^2$ throwing one knot onto another. We call $\kappa(L)$ the knotification of L.

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Theorem (Hedden-K.)

If a $L \subset S^3$ is an n-component link with first non-vanishing $\overline{\mu}_L(I)$ invariant weight $rn + 1$, then $D(\kappa(L)) \geq r + 1$.

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Thank you!